Uncovering the Analytical Saffman-Taylor Finger in Unstable Viscous Fingering and Diffusion-Limited Aggregation

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Very unstable viscous fingers moving in a linear channel are investigated as well as diffusion-limited aggregates grown in a strip between two reflecting walls. In both cases a large number of independent runs are performed and the cell occupancy distribution is measured. It is shown that the zone of large occupancy has the width and shape of the Saffman-Taylor finger $\lambda = 0.5$. Similarly, in sector-shaped cells, the width of the large-occupancy region is the limit width of the stable fingers. In the case of a 90° cell, its shape is that of a predicted analytical solution.

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Diffusion-controlled growth has been the subject of much recent interest.[1,2] In the classical areas of dendritic growth, viscous fingering in diphasic fluid flow, and flame propagation, the steady-state motion is now rather well understood.[2] The unsteady regime seems to be well described by various models of stochastic growth initially proposed for fractal aggregates.[3,4] We will limit ourselves here to the two simplest cases: viscous fingering in an isotropic Newtonian fluid in a classical Hele-Shaw cell and Witten-Sander's diffusion-limited-aggregation (DLA) model.[5] The similarity of their equations is well known.[3] In both cases the growth takes place in a Laplacian field (pressure for viscous fingering and walker's probability of visit for DLA) and the growth velocity of the pattern is proportional to the local gradient of this field. The difference between the two descriptions comes of course from their deterministic or stochastic nature but also from the existence of surface tension in viscous fingering which has no obvious counterpart in DLA. Several experiments[6,7] and numerical simulations[3(b),4,8] have been devoted to the comparison of the patterns obtained in these two types of growth. The aim of the present Letter is to investigate the statistical properties of very unstable fingers and DLA aggregates in specific geometries. The choice of the geometry is crucial because in Laplacian pattern-forming systems the motion of one of the boundaries (the interface) depends on the other boundary conditions fixed by the cell's shape. Three main configurations have been considered: (i) In their original work, Saffman and Taylor[9] used a long linear channel of width W and this corresponds to growing DLA clusters in a strip geometry;[10] (ii) a radial configuration has been studied both for viscous fingering[11] and for DLA[5] (where it is the most usual choice); (iii) a third intermediate situation was introduced recently,[12] in which the cell has the shape of a sector of a disk. We limit ourselves here to the investigation of unstable fingers and DLA aggregates in *those configurations in which the experimentally stable Saffman-Taylor solutions are known:* the linear and the sector-shaped geometry.

Let us first recall the main results on stable fingers. The experimental situation in linear cells is defined by only one parameter:[3(b)]

$$1/B = 12(\mu V/T)(W/b)^2 \approx 118(W/l_c)^2,$$

where W is the cell width, b its thickness, μ the viscosity of the most viscous fluid, T the surface tension, V the interface velocity, and $l_c = \pi b(T/\mu V)^{1/2}$ the capillary length scale. For $1/B < 7000$ a single finger is observed, scaled on the cell width. In the upper part of this stability range the relative width λ of the finger tends asymptotically towards 0.5. Saffman and Taylor[9] had found a family of solutions for the interface shape at $T=0$. It is only recently[13] that the observed asymptotic finger width $\lambda = 0.5$ was understood. It results from the selective action of surface tension acting as a singular perturbation. Somewhat similar results were obtained experimentally in sector-shaped cells[12] of angle θ_0. Here $1/B$ is a local parameter where W is $\theta_0 r_0$ (r_0 is the distance of the finger tip to the apex). The finger can move either from the apex to the periphery (divergent finger, $\theta_0 > 0$), or from the periphery to the apex (convergent finger, $\theta_0 < 0$). For all convergent fingers and for divergent fingers up to $\theta_0 < 20°$, an asymptotic minimum angular relative width λ_m was found to depend linearly on the cell angle.[12,14] At large $1/B$ values, in parallel cells as well as in the sector ones, the fingers are unstable and have a fractal-like appearance. In this Letter we present a statistical analysis of these very unstable viscous fingers and of DLA clusters. We show that their mean occupancy distributions are very similar and reflect the structure of the above-mentioned stable solutions.

Our experimental setup was designed to reach very large values of $1/B$ which implies large applied pressures. We used a linear channel and four sector-shaped cells ($\theta_0 < 90°$). All were made of float glass 19 mm thick. The cell thicknesses b were either 0.25 or 0.125 mm and their width in the region of measurement was $W = 10$ cm. We used a silicon oil, Rhodorsyl 47 V 500, with

$T=21\times10^{-3}$ N/m and $\mu=0.48$ kg/ms. In both the linear and the divergent cells air was injected at a controlled pressure, and for convergent fingers oil was pumped out of the apex and an experimental expedient was used: At rest we created a dip in the meniscus at the periphery of the cell so that the finger would originate from this point on the cell axis. (The same expedient had been used[12] for stable fingers.) A typical unstable viscous finger grown in a linear cell is shown in Fig. 1(a). It is compared to DLA clusters [Fig. 1(b)], computed in strip geometries of width $W=32$, 64, or 128 lattice units. We used an on-square lattice algorithm and reflecting lateral walls.[15] As in the experiments,[12] a numerical trick was used to initiate the growth: The very first random-walking particle sticks on a needle a few particles long centered on the cell axis. Similar DLA simulations were conducted in a sector geometry (the cell boundaries are approximated by a staircase structure and the particles are locally reflected normally to these boundaries). In both experiment and simulation we wanted, after a large number of independent runs, to measure the mean occupancy of each site of the channel. The most extensive analysis was carried out in the DLA case. In a given strip we grew N aggregates with the same total number M of particles. We then counted for each point of the grid how many times it had been occupied by a particle

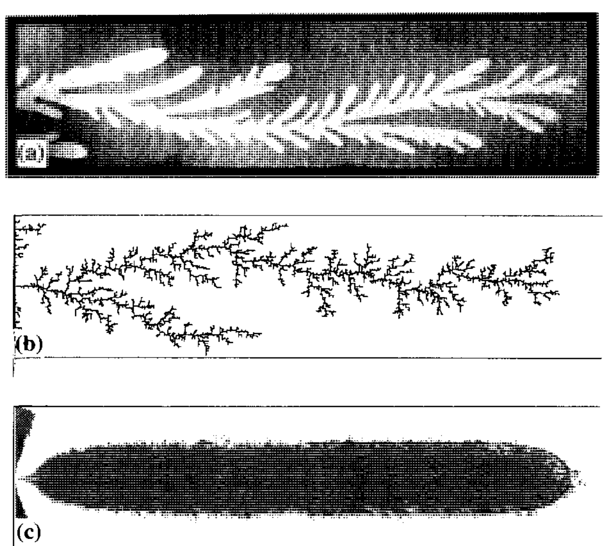

FIG. 1. (a) Photograph of a very unstable Saffman-Taylor finger in a linear Hele-Shaw cell of width $W=10$ cm and thickness $b=0.25$ mm at $1/B=3.3\times10^5$. (b) A DLA cluster of 6000 particles grown in a strip of width $W=128$. (c) The points of the cell where the occupancy rate is larger than $r_{max}/2$ are represented in gray. This repartition was obtained from the analysis of 510 aggregates of the type shown in (b) and having the same mass $M=6000$. The continuous line is the shape of the Saffman-Taylor analytical solution of width $\lambda=0.5$.

of an aggregate. This number, divided by N, gave $r(x,y)$, the mean occupancy of this point. In viscous fingering our smallest scale is larger than in DLA so we did not need such a large number of runs; N ranged from 50 to 120. We limited ourselves to the measurement of the mean occupancy across the cell. We chose a section of the cell (i.e., an arc of circle for sector-shaped cells) where the pattern had finished its evolution. We built a histogram of the occupancy by air in all the runs. A division by N gave the mean occupancy r in this section.

The histogram of the occupancy along the axis of the linear strip obtained from the DLA simulations shows that, except in the initial region and in the tip region, r is constant. This means that in the regions where the growth has ceased the cell translational invariance imposes itself on the occupancy profile. We grew aggregates with M from 1000 to 6000 in strips of $W=32$, 64, and 128. Scaled on W, the mean length of the cluster x_t/W is proportional to its mass. The falloff of r in the tip region, of width Δx_t, corresponds to the active part of each pattern and to the dispersion of the tip position. Δx_t is also scaled on W and increases slowly[16] with x_t. Across the linear cell, all transverse occupancy profiles have a maximum value r_{max} at the center ($y=0$) and decrease to zero at the walls ($y=\pm W/2$). For viscous fingers, the profiles (a sharp step profile for stable fingers) become smoother with increasing $1/B$ and the width of the regions which are never visited (along the walls) reduces. The limiting profile of the histogram is obtained for DLA, where it is surprisingly well fitted by $r(x,y)=r_{max}\cos^2(\pi y/W)$. The fundamental result about all these histograms is that *their width at midheight is half the channel width* $\lambda=0.5$. In other terms, going from the stable finger to unstable patterns, the occupancy rate becomes smeared out but its width at midheight is preserved. The results are in fact even more specific: Figure 1(c) shows all the points of the strip where r is larger that $r_{max}/2$ in a series of 510 DLA simulations. The limit of this region is very well fitted by the Saffman-Taylor analytical solution[9] for $\lambda=0.5$. We checked that other sections at different levels of occupancy have different widths but are not fitted by other Saffman-Taylor solutions. This result is strongly enforced by our experiments in sector-shaped cells. In each case the transverse histograms (measured on an arc of circle) have a similar shape for DLA and viscous fingering. For convergent patterns the peak is narrow and there is a large region on each side where $r=0$. For divergent patterns the histogram has a large flat plateau at r_{max} and falls rather abruptly to zero. For all angles $-90° < \theta_0 < 90°$, the width at midheight is best fitted by a linear relation (Fig. 2):

$$\lambda_m=0.5+(3.4\pm0.2)\times10^{-3}\theta_0 \tag{1}$$

(with θ_0 in degrees). It is remarkable that for all convergent channels and for the divergent ones with $\theta_0<20°$,

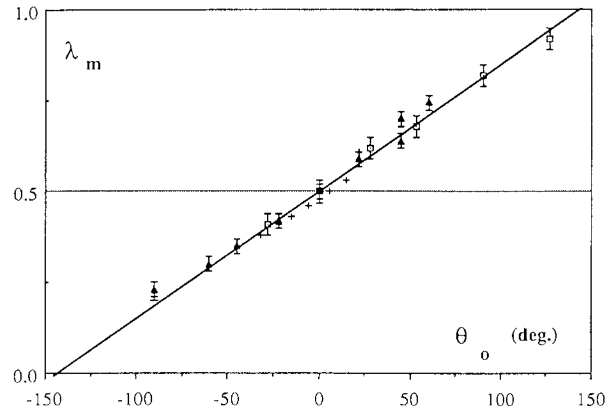

FIG. 2. Width of the region of large occupancy in sector-shaped cells as a function of the cell angle θ_0. ▲, unstable Saffman-Taylor fingers; □, DLA aggregates; +, minimum width observed for stable fingers in these cells; and ——, best linear fit [relation (1)].

λ_m has the same value which characterized the asymptotic width of stable fingers.[12] For $\theta_0 > 20°$ the same identity is not observed because, as noted in Ref. 12, the stable fingers destabilized before reaching their asymptotic width. If one keeps extrapolating relation (1) for larger θ_0 values, the width at midheight of the histogram of the transverse occupancy distribution is expected to fill the entire cell width for a critical value of the sector-cell angle: $\theta_0 \approx 4\pi/5$. Our preliminary results suggest that this critical angle marks a transition in the shape of the mean occupancy profile which then presents several lobes. The presence of a fivefold symmetry in diffusion-limited aggregation has already been suggested in previous works[17-19] and we will elaborate on this point in a forthcoming publication.

Divergent cells with $\theta_0 = 90°$ are of particular interest because a family of analytical solutions is known,[12] giving the finger shape in the absence of surface tension. These fingers form a self-similar counterpart to the Saffman-Taylor solutions. Their equations, also parametrized by their relative angular width λ, can be written[12]

$$x = \frac{\sqrt{2}}{2}\cos\alpha[(\tan\alpha)^a + (\tan\alpha)^{1-a}],$$
$$y = \frac{\sqrt{2}}{2}\cos\alpha[(\tan\alpha)^{1-a} - (\tan\alpha)^a]\tan\left(\lambda\frac{\pi}{4}\right), \quad (2)$$

where $\lambda \in [0,1]$, $a = (1-\lambda)/2$, $\alpha \in [0,\pi/2]$, and Ox is along the finger axis. Figure 3 shows two realizations of unstable patterns and the region of large occupancy. The angular width of the latter is $\lambda_m = 0.82 \pm 0.02$. Its shape is very well fitted by the corresponding solution of Eq. (2) and differs markedly from the profile that would be obtained from the conformal transformation of the Saffman-Taylor fingers.[12]

On the same line, we found for DLA clusters as well as viscous fingers that the mean occupancy of the cell width varies as $(1/B)^{-0.19}$. The interpretation of this scaling law[20] is clear. The elementary length is the capillary length l_c; the pattern can be considered as a fractal of dimension d_f from l_c up to W the width of the cell. The area occupied by the pattern in a square $W \times W$ scales like $l_c^2(W/l_c)^{d_f}$. Therefore the mean density scales like $(W/l_c)^{d_f-2} \sim (1/B)^{(d_f-2)/2}$. The observed power-law dependence, with exponent -0.19 ± 0.01, corresponds to $d_f = 1.62 \pm 0.02$ which is in good agreement with the fractal dimension measured both for viscous fingers[6(b)] and for DLA patterns[4,17] in circular geometry.

In conclusion, our results confirm once more[3,4,6] the similarity between the fractal structure of viscous fingers and DLA patterns. This suggests that the detailed noise mechanism does not play an important role. Because our experiments and simulations were done in geometries where stable solutions are known, they bring new results. It is not such a surprise that the cell geometry should determine the large-scale shape of the profile of mean occupancy. It is a very striking result, however, that the selected solution should be the same as the stable one.

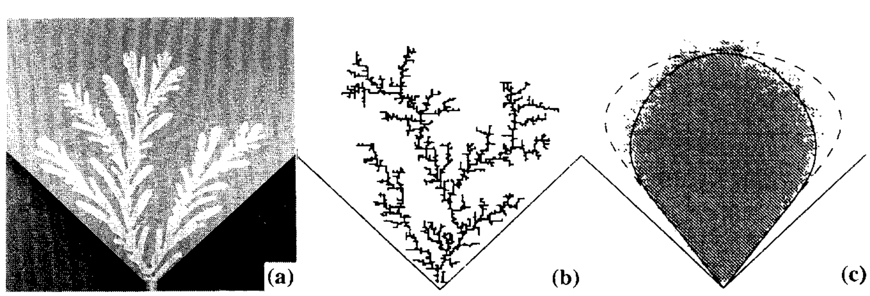

FIG. 3. (a) Unstable viscous finger and (b) DLA cluster grown in a cell of angle $\theta_0 = 90°$. (c) In gray, region of occupancy rate $r > r_{max}/2$ for $N = 420$ DLA clusters of mass $M = 2000$. The continuous line is the analytical solution given by Eq. (2) for $\lambda = 0.82$. The dashed line is the conformal transform of the Saffman-Taylor solution $\lambda = 0.82$.

In sector-shaped cells, when the structure diverges from the apex, it builds up a fractal structure in a larger and larger range of scales between l_c and $\theta_0 r_0$. During this buildup it retains the same sensitivity to both the large and the small scales. In other terms the selective action of the microscopic length scale acts through the entire range, up to the largest scale of the pattern. In that respect our results bring the clue that the role of the viscous-finger capillary length scale is played by the lattice-mesh size for DLA clusters. This suggests furthermore that the mean density profile is described by effective equations similar to the ones describing smooth stationary states. This phenomenon would explain naturally the well-known sensitivity[5] of the shape of large DLA clusters to the underlying lattice anisotropy. In this case, the corresponding smooth stationary states are the parabolic needle crystals whose very existence is dependent on anisotropy,[2] once surface tension effects are taken into account. We are currently studying this analogy. What are the effective equations describing the mean density profile? Two natural candidates would be the mean-field equations for DLA[21] or the Saffman-Taylor equations[9] themselves. The first one does not give a density profile independent of the position along the cell's axis in the trailing part of the finger and the second possibility is not compatible with our results as it would require that, for large statistics, the mean density profile approaches a sharp step profile. More work is needed to elucidate this question. *In fine,* we would like to suggest that the appearance of simple large-scale coherent structures can show up in the buildup of many other fractal structures if an adequate type of averaging is performed.

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[16]Though our results are not very precise, they are consistent with a dependence $\Delta x_t/W \propto (x_t/W)^{0.5}$. This is explained naturally if the growth process, on the scale of W, can be considered as the successive addition of n independent bunches of a fixed number of particles. These, having different configurations, have different lengths, and so the dispersion of the tip positions is proportional to $\sqrt{n}$.

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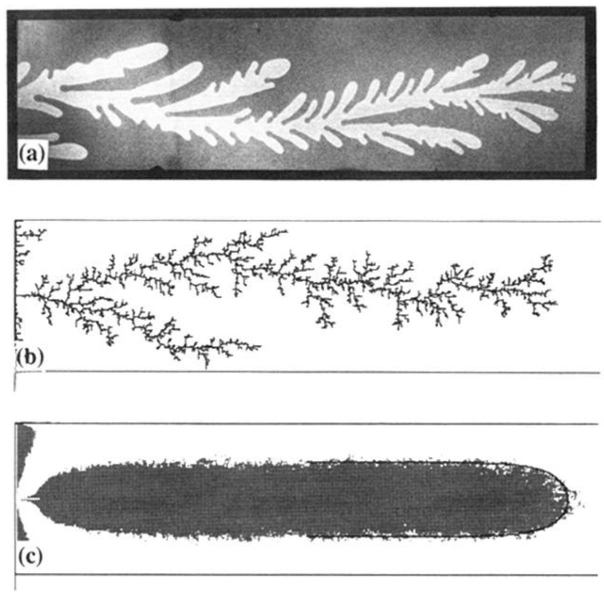

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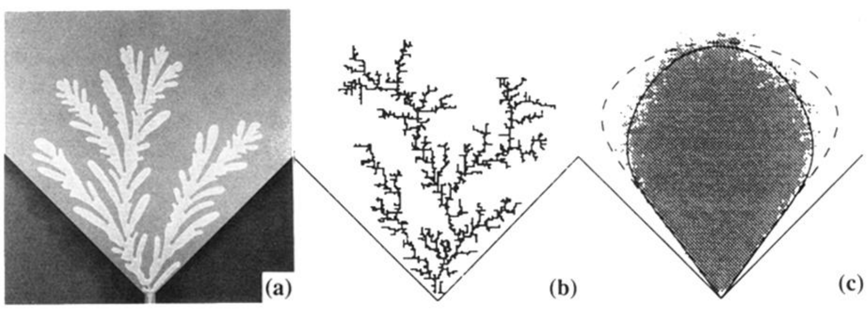

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