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Optimal routing in sailing

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Abstract

This paper discusses how to determine the best trajectory for a sailing boat and draws some analogies between this routing problem and condensed matter physics or geometrical optics.

1 Introduction

In many sports the problem of the choice of the best track is not relevant, for example if you run a 400 meter in a stadium or if you run a marathon. In other sports, as downhill skiing for example, there is more freedom in the choice of your trajectory but *in fine* there is little difference between trajectories of all the skiers. However, in long distance sailing, e.g. a transatlantic race or an around the world race, trajectories are pretty free and you can win a race choosing a longer path. Indeed the shortest path is usually not the quicker one and weather conditions can impose you a much longer path. Minimizing time of sailing from point A to B,

$$T_{AB} = \int_A^B dt = \int_A^B \frac{dl}{v} \tag{1}$$

instead of minimizing the distance

$$L_{AB} = \int_A^B dl, \tag{2}$$

is clearly the key to win such long distance race.

This is clearly illustrated in figure 1 which shows the boat trajectories during the 2016 transatlantic sailing race form Plymouth to New-York. François Gabart won the race although his trajectory, the most southward one, was in fact the longest one: 50 % longer than the orthodromic distance. He chooses this long trajectory to find better oriented winds to exploit the downwind capacities of his multihull *MACIF*.

2 Speed diagram of a sailing boat

In order to compute the least-time trajectory we first need to know the possible velocities of our sailing boat. Depending on wind direction, wind intensity, selected sails, wave amplitudes or water currents, target velocities can be computed by naval architects before boat construction or measured in details at sea. Figure 2 presents, in polar coordinates, an example of such target velocities for a given wind for each sailing direction θ with respect to the true wind. Boats being usually left/right symmetric, only a half of the curves are represented (port tack). Here the two color curves correspond to two different front sails (a genoa and a spinnaker). These speed diagrams are not isotropic, they present directions of lower velocities for some specific angle ranges, and there are not even convex

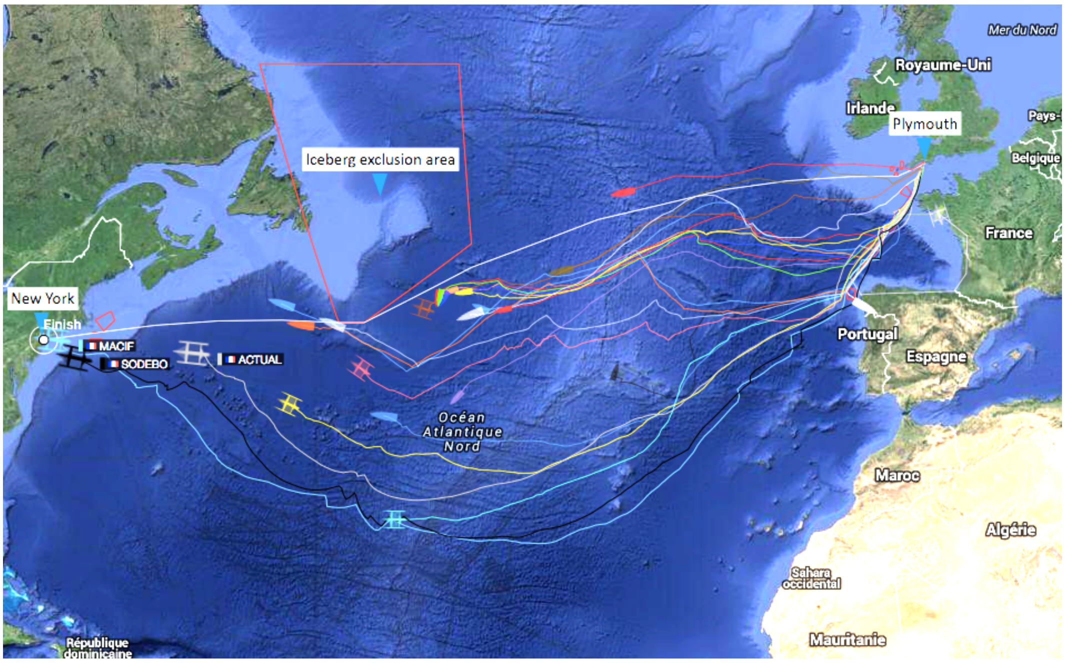

Figure 1: Positions and trajectories of the participants of ”The Transat Bakerly” in May 2016 when François Gabart reached New-York with his trimaran MACIF. His trajectory is the longest one (the southern one, in blue). Source http://www.thetransat.com.

functions. For example it is impossible to sail directly against the wind (at small θ). It is however possible to complete the speed diagram by its convex envelope, as done in figure 2 with the three black lines [1]. These straight lines correspond to achievable effective velocities when taking into account the possibility for the boat to tack, i.e. to make some zig-zags. The two horizontal lines correspond to the best upwind or downwind effective velocities when tacking, the so-called Velocity Made Good (VMG). The third oblique line corresponds to the possibility to use alternatively the genoa or the spinnaker at the optimal sailing angles indicated by the black dots.

3 Analogy with the growth of a crystal

The speed diagram of a sailing boat and the construction of its convex envelope presents a strong analogy with the Wulff construction used to predict the shape of a crystal in condensed matter [2]. Indeed in 1901, G. Wulff [3] proposed a method to find the equilibrium shape of a crystal based on the plot of the Gibbs free energy $\gamma(\theta)$ as a function of crystalline directions. Figure 3 presents such function for a four-fold crystal. The plot of the perpendicular to each vector $\gamma(\theta)$ gives the corresponding Wulff plane and the *inner* envelope of all the Wulff planes corresponds to the crystal shape, here the blue square. Since $\gamma(\theta)$ represents the inverse of the crystal growth speed in the direction θ whereas $V(\theta)$ is directly the boat velocity, in the Wulff construction it is the inner envelope of the normal to the vectors that are plotted whereas in the speed diagram it is the outer envelope of the tangents to the curve.

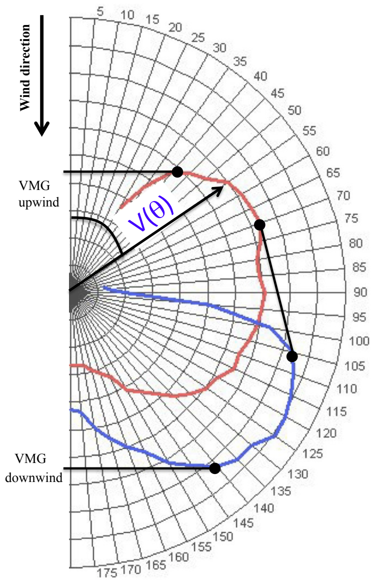

Figure 2: Speed diagram of a sailing boat for a given wind intensity and sea state. This polar curve represents the best velocity $V(\theta)$ of the boat when sailing at angle θ with the direction of the incoming true wind. The red and blue curves correspond to different front sails, and the three black lines correspond to the convex envelope of the speed diagram.

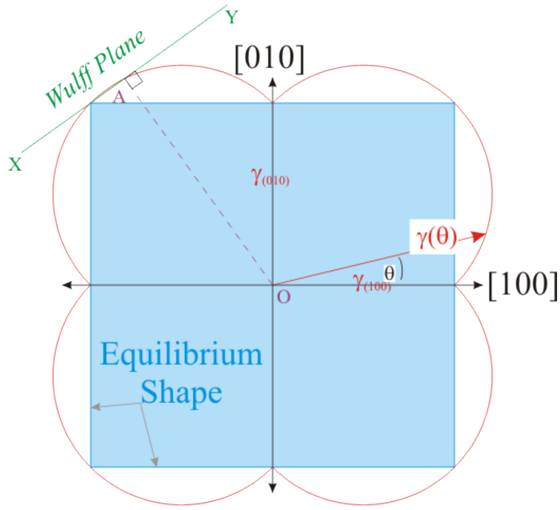

Figure 3: Wulff construction (from Wikipedia) giving the equilibrium shape of a crystal as the inner envelope of the polar surface energy $\gamma(\theta)$.

4 Method of isochrones

The knowledge of the speed diagram is the first step to determine the optimal trajectory. We will now describe the method to find the least time trajectory. We will however make first a strong simplification: we assume that the weather forecast is perfectly known, i.e. we know for certain the wind direction and the wind intensity $\mathbf{W}(\mathbf{r}, t)$ at any location $\mathbf{r}$ and at any time t. This is of course a crude approximation of the reality and we will revisit this hypothesis in the conclusion.

The method of isochrones is the most widely used method to calculate the best trajectory between two points. It is based on the determination at any time of the curve of the largest distances reachable in any direction from the initial position of the boat. For example suppose at $t = 0$ that the boat is in A. The knowledge of the wind vector there and of the convex envelope of the speed diagram give the reachable positions in any direction θ after a given small time interval Δt : $\mathbf{r} = \mathbf{V}(\theta)\Delta t$. The ensemble of all the reachable positions for all θ defines the first isochrone curve at time Δt and it has thus the shape of the convex envelope. Note that the time interval must be small enough to be able to neglect time or spatial variations of the predicted weather. After this first iteration the process is repeated: at each position $\mathbf{r}$ along the previously determined isochrone, the polar boat speed diagram corresponding to wind intensity $W(\mathbf{r}, t + \Delta t)$ is drawn and oriented in the wind direction. The new reachable positions after Δt can thus be computed again. Taking the outer envelope of all the new reachable positions gives the new isochrone at time $t + 2\Delta t$. The process is repeated until the time t_{finish} when the last isochrone crosses the finish line at point B. The minimum-time path from point A to B is then determined back in time as the trajectory that connect the corresponding points on each isochrone. Note that the trajectory is normal to an isochrone only if the speed diagram presents a maximum along that direction. More details on the isochrone method, and on the process of minimizing the number of tacks can be found in Ref. [4, 5].

The isochrone method can be easily implemented on a computer. Various softwares are available and are currently used before or during the race. You first need to incorporate the boat speed diagrams, to specify the boat current position and the finish point B and to download the latest weather forecasts for the sailing area. In less than a minute the isochrones are computed and the best trajectory is visualized on the map. Figure 4 presents the result of such simulation with the free routing software qtVlm[1].

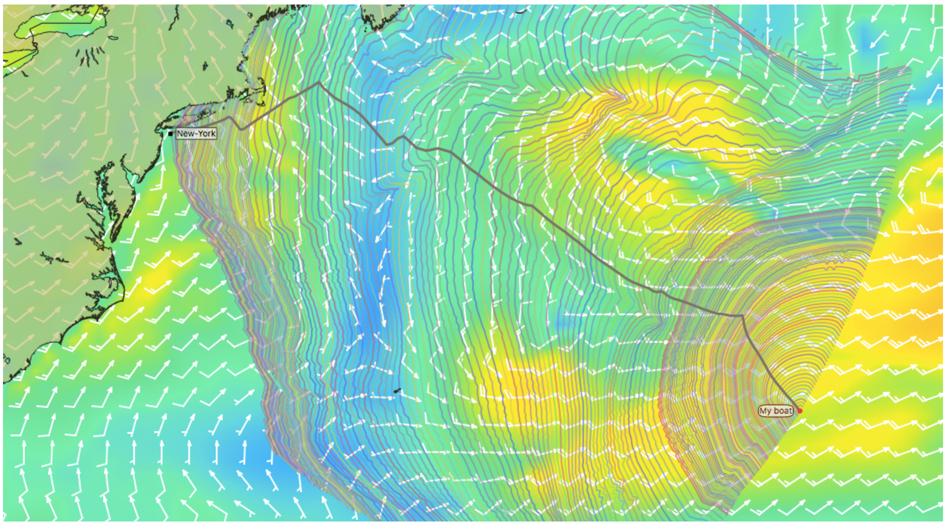

Figure 4: Result of a simulation by the method of isochrones using the software qtVlm for a navigation starting from an initial position in the middle of the Atlantic up to New-York. Background colors and vectors correspond to the initial weather prediction. The curves are the computed isochrones. The optimal trajectory is the thick line that crosses the isochrones. This trajectory presents a number of cusps that correspond to upwind or downwind tacks.

5 Analogy with geometrical optics

The optimal routing determination presents a strong analogy with geometrical optics and this analogy was first described in details by Kimball & Story in a very stimulating paper [6, 7]. Indeed, as for optics, the optimal sailing trajectory corresponds not to the one that minimize the length L_{AB} of the trajectory (Eq. 2), but to the one that minimizes the duration T_{AB} (Eq. 1). This minimisation principle is exactly the Fermat principle of least time. In 1637, Pierre de Fermat was indeed able to recover the Snell-Descartes law of refraction at the interface between two mediums, assuming that light selects the trajectory of minimum duration.

[1]This software was initially developed for the virtual sailing community (http://wiki.v-l-m.org/index.php?title=QtVlm/en). Commercial softwares as Adrena (http://www.adrena.fr/en/) or Maxsea (http://www.maxsea.com) are largely used by sailors.

The isochrones of a sailing boat are analogous to the wavefronts of geometrical optics, the speed diagram playing the role of the inverse of the refractive index. The isochrone construction method corresponds to the Huygens construction for light wavefronts, where each point of a wavefront is considered as a source of circular waves (figure 5).

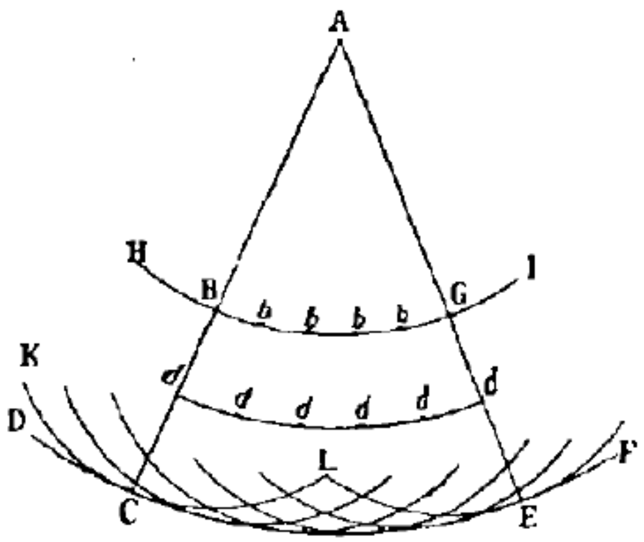

Figure 5: Drawing of Christian Huygens in his "Traité de la lumière" (1690). Each point of a wavefront at time t is a source of circular waves. The envelop of all the circular waves gives the new wavefront at time $t + \Delta t$.

As the refractive index of usual material is a scalar field $n(\mathbf{r})$, isotropic and time independent, Huygens construction to determine the light path is simpler than the determination of the best route for a sailing boat, because the speed diagram of a boat is direction and time dependent. When the refractive index depends on position, as in a stratified fluid for example, curved beams can be observed, and explains the origin of mirages. In some particular crystals the speed of light could be a non-isotropic function, and the concept of index ellipsoid replaces then the usual scalar refractive index: the speed of light diagram is no more a sphere but an ellipsoid (and thus remains a convex function). Note however that even in that case the symmetry $c(\theta) = c(\pi + \theta)$ still holds for light and the optical path from A to B is identical to the reverse path from B to A. This principe of reversibility of light does not hold in sailing, even with constant weather. Huygens construction for sailing boats remains a subtil subject [6].

6 Conclusion

Routing of sailing boats presents some interesting analogies with other domains of physics. However finding the best trajectory on the sea could be more complex than presented here. Indeed the crew is not always able to sail the boat to its target velocities, or these target velocities are not perfectly known, and more important the weather could be different than the meteorological forecasts. Because of the chaotic dynamics of the atmosphere, the computed trajectory becomes certainly not optimal for race longer than a few days. The determination of an index of confidence for the optimisation results remains nowadays a difficult task.

The art of a good weather router involves thus a lot of experiences and a thoughtful analysis of the stochastic nature of meteorological predictions. This can be tested either by running various simulations with weather forecasts coming from different agencies, or adding some stochastic noise on the data [8, 9] in order to determine not only the fastest trajectory, but also the most robust and safer one. Moreover, the sailor can sometime chose to give up with the optimal trajectory for safety reason or for tactical reasons: e.g. it can be opportune for the race leader to relocate his boat between the second boat and the finish line, in order to control it in case of a possible unpredicted evolution of the weather.

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