Hierarchical crack pattern as formed by successive domain divisions
Part I: Temporal and geometrical hierarchy

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Crack patterns, as they can be observed in the glaze of ceramics or in desiccated mud layers, are formed by successive fractures and divide the two-dimensional plane into distinct domains. On the basis of experimental observation, we develop a description of the geometrical structure of these hierarchical networks. In particular, we show that the essential feature of such a structure can be represented by a genealogical tree of successive domain divisions. This approach allows for a detailed discussion of the relationship between the formation process and the geometric result. We show that -with some restraints- it is possible to reconstruct the history of the system from the geometry of the final pattern.

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I. INTRODUCTION

A large variety of morphologies of crack patterns can be found in nature. In many cases, the fractures form a closed network and thus divide a two-dimensional surface into distinct domains. Some morphologies show an astonishing similarity to two-dimensional soap foams which have often been considered to be the model system for space-dividing pattern [1, 2]. For instance, the transverse section of a basalt formation, supposedly formed by a mechanism called columnar cracking, is composed of hexagon like domains [3–5] and show statistical properties similar to foams. The fractures propagate simultaneously into the volume and interact symmetrically; there is no apparent hierarchy in the resulting structure. Another case of crack patterns similar to foams was observed in ceramic disks subjected to rapid thermal shocks. Again, the fractures can be considered as simultaneous [6].

There is, however, a different crack morphology with an accentuated hierarchy. It is the result of the shrinking of material layer frustrated by its deposition on a non shrinking substrate. Such patterns are observed in the glaze of ceramics or in desiccating mud or gel. Studies in coffee ground [7] and in desiccating colloidal soils [8] revealed that where the material layer is not too thin, the fractures are formed successively and each new fracture joins older fractures at each extremity. The result is a space-dividing pattern showing a strong hierarchy of fractures of different lengths. Both studies reveal furthermore that the characteristic length scale of the final pattern (domain size) scales linearly with the layer thickness. In this paper we investigate this regime, mainly because it can be considered as a physical model system for hierarchical space divisions in general. Other examples of hierarchical space division include the venation pattern [9] in vegetal leaves or the partitioning of a city into blocks.

In contrast to the regime of very thin layers, where the nucleation and propagation of the fractures are dominated by material heterogeneity, and which has been widely studied numerically (see for instance [10–14]), no theoretical framework for the hierarchical regime exists. On the basis of an experimental example, we will work out an appropriate description of the hierarchical crack pattern. This description is directly based on the hierarchy and the space diving property as the main features of the system. It allows a detailed analysis of the relation between the geometrical structure and the history of the system. In particular, we discuss to what extent the pattern's history can be reconstructed, knowing only its final geometry. The accented geometrical heirarchy emphasizes the importance of history.

The concept being introduced defines furthermore the framework that we will use in the in the second part of the paper [18] for a detailed study of the evolution of the domain shapes.

II. EXPERIMENT

We modified an experimental system that has already been used to study the cracks in droplets [15] or directional cracking [16]. We fill a shallow, circular container (diameter $\sim$ 10mm, height $\sim$ 0.5mm) with an aqueous solution of latex particles (diameter $\sim 0.1\mu m$). The bottom of the container is a clean glass plate, the lateral walls are altuglass. The contact line of the solution is quenched at the upper edge of the circular wall and remains there during the whole experiment. In this way we obtain a layer of approximately constant thickness in the center of the container, and avoid the anisotropy

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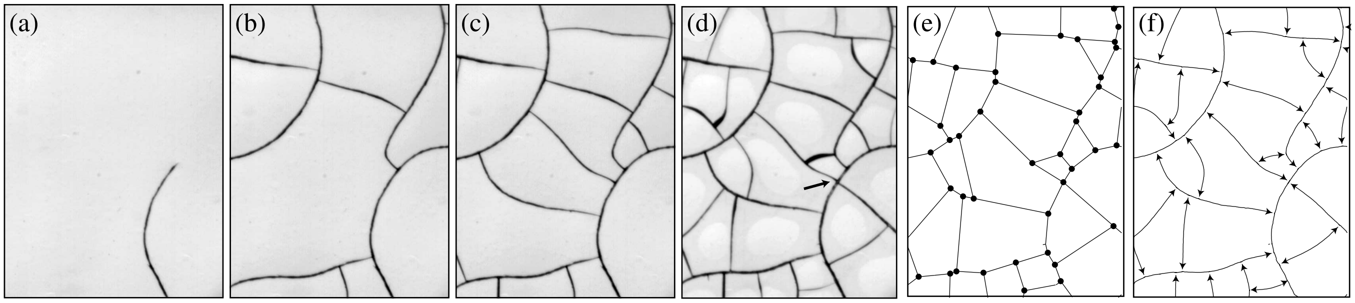

FIG. 1: (a-d) Photographs of the formation of a crack pattern. (e) The representation of the final pattern (d) as an embedded graph. The disks represent the nodes, the lines the edges. (f) The reconstructed cracks. The arrow heads indicate the geometrical hierarchy relation between them.

due to large evaporation at the borders.
As water evaporates, the concentration of the solution increases. When it exceeds a critical value, the material becomes a gel that adheres to the glass plate (the substrate). Further evaporation induces a shrinkage of the gel layer, but the adhesion to the solid substrate limits the contraction of the gel. This frustration causes mechanical tensions that are relaxed by the formation of fractures. Figure 1(a-d) shows a selection of photographs of the formation of the crack network. The photos are limited by the camera field; the borders of the image have no physical meaning.

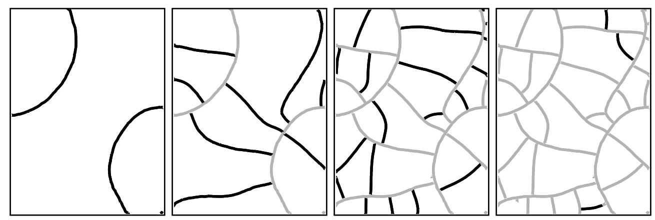

FIG. 2: The geometrical orders of the cracks. From left to right: First , second, third and fourth order. The cracks of lower orders are drawn in gray.

III. HIERARCHY

The cracks are formed successively and, using the video sequence, we attribute to them a temporal order: first, second and so on. In the shown experiment we observe 28 successive cracks. We shall call the succession of the cracks their temporal hierarchy.
The temporal hierarchy of the cracks is of crucial importance because the effect of one crack on another is not symmetrical. A crack remains unchanged after being formed and is therefore not affected by cracks formed later. In reverse, the existing cracks define the boundary conditions for the mechanical stress field that governs the formation of the future cracks. In particular, a crack of higher temporal order (younger) joins a crack of lower temporal order at an angle close to 90°.
From an abstract view-point we can consider the final crack pattern (fig. 1(d)) as the two-dimensional embedding of a graph e.g. as a set of nodes and edges (fig. 1(e)). This approach has been found useful in the case of soap foams. The graph represents the topology of the pattern and can be detected by traditional image processing tools. In the case of the crack pattern, however, this representation is somewhat artificial since the fractures are decomposed into parts of fractures. In order to reconstruct the continuous fractures we can take into account the angles at each node : the two edges that form locally an angle close to 180° belong to the same fracture. In practice, we used an image processing that has been developed for a different purpose [17] and that detects the topology automatically and measures the local angles. Using the 180° criterion, we paste the edges together and obtain the fractures as continuous lines (fig. 1(f)). One can verify that these lines actually correspond to the cracks as they were observed during the formation process.

Cracks of higher temporal order join cracks of lower temporal order with $\sim 90°$. We can consider inversely this property at the nodes as the indicator of a geometrical hierarchy between the cracks. They are indicated by the arrowheads in fig. 1(f). Using these local relations, we define global geometrical orders to each fracture by recursion. Cracks of geometrical order one do not connect to any other crack; their extremities are outside of the observation window. Cracks which end on first order cracks are called of second order. In general, a crack of order n ends, at least at one of its extremities at a crack of order $n-1$. In this way, we obtain in our example cracks of four distinct geometrical orders. They are shown in fig. 2.
Compared to the 28 temporal orders, the number of detected geometrical orders is lower. Furthermore it may occur that a crack of higher geometrical order is older than a crack with a lower temporal order. It is thus not possible to reconstruct the temporal succession of the cracks by the means of the geometrical criteria used. The origin of this apparently disappointing fact is never-

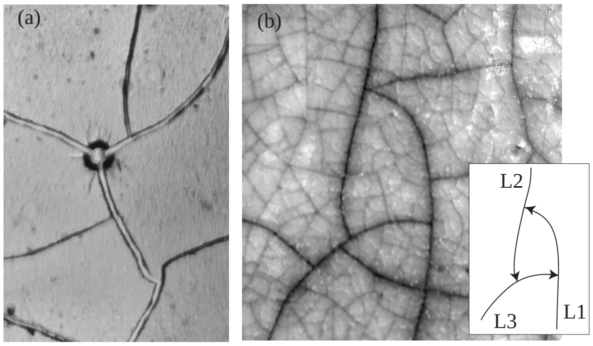

FIG. 3: Two cases where the introduced description must fail. (a) A triplet of cracks is formed at a defect of the layer. (b) Three fractures form a loop.

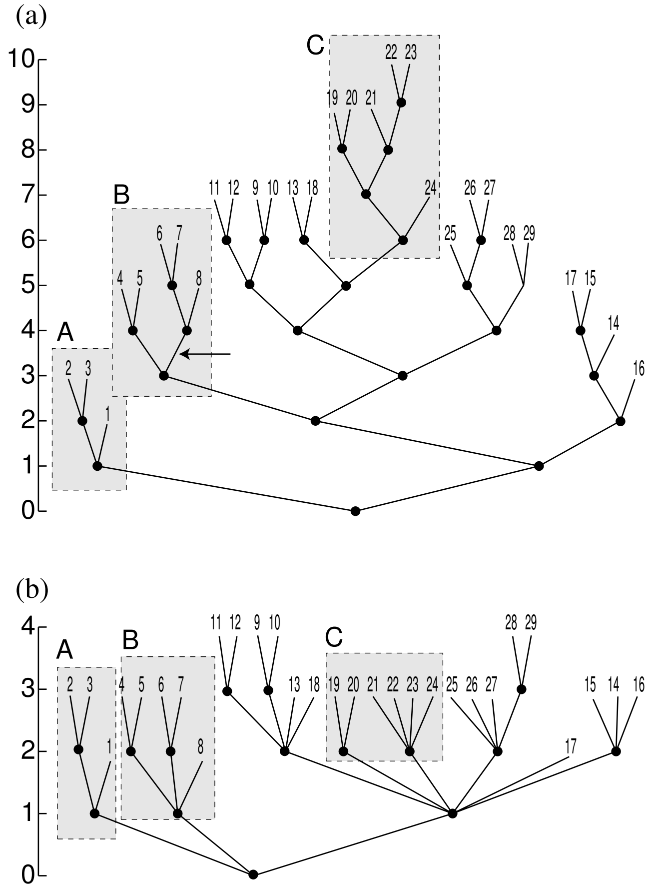

FIG. 4: The genealogical trees of domain division: (a) established on the basis of the real temporal evolution, (b) established on the basis of the geometrical hierarchy. The generations of the domains are indicated on the left. The black disks represent intermediate domains, the numbers represent the undivided domains as they appear in the final pattern.

theless the physics of the system. A fracture divides the entire layer thickness. Since the substrate can be considered as infinitely rigid, the layer on one side of the fracture side is then entirely decoupled from the layer on the other side. Since the fractures formed in different domains are independent, there exist no geometrical criteria to reconstruct their successive order. The lower number of temporal orders of cracks reflects this independence. This observation is the basis of the following section.

The hierarchical order of the fractures is paired with their succession, which is particular to this cracking regime. This is illustrated by the counter examples shown in fig. 3 where the crack orders are undefined. In fig. 3(a), three cracks nucleate in a star like formation with relative angles of 120°. These nucleations occur mainly at materiel defects in very thin layers (see [7, 8]) and there is neither a temporal succession of the cracks nor a local geometrical hierarchy. In fig. 3(b), three cracks form a turning loop in the glaze of ceramics. According to our analysis, the fracture L1 should be younger than the fracture L2 which is younger than L3, itself younger than L1. We do not understand the origin of this paradox, the cracks in the glaze of ceramics propagate very rapidly and their nucleation is very sparse. Although we searched for further examples in all ceramics observed over a period of one year, we found none. The image thus presents a very particular exemption, at least in the glaze of ceramics. However, the failure of the concept of geometrical orders is in both cases due to a different physical regime, where the fractures are not formed in strict succession.

IV. THE GENEALOGICAL TREE OF DOMAIN DIVISION

A. Construction of the trees

The crack pattern becomes conceptually simpler if we focus on the domains as the relevant entities rather than the fractures. Since each crack is connected at its extremities to other cracks, the resulting pattern is a space dividing structure. We will call a domain each island that is limited by cracks. A new crack always divides one domain into two. In analogy to the biological cell division, we call them respectively mother and daughter domains. Since the daughter domains are mechanically separated, cracks formed in different domains are independent from one another. The way a domain is divided depends only on the domain itself and not on its neighbors.

Since there is no use to relate cracks in different domains, we should find a representation of the structure that takes the separation in non-interacting sub-systems (domains) into account. Let us therefore introduce a representation that we will call the genealogical tree of domain division. We will distinguish between the tree constructed on the basis of the temporal succession of the cracks (temporal tree, fig. 4(a)) from the tree constructed on the basis of the geometrical hierarchy (geometrical tree, fig. 4(b)).

Let us first consider the temporal tree in fig. 4(a). Initially there is one, non-divided domain. We attribute the generation zero to this domain and represent it as a

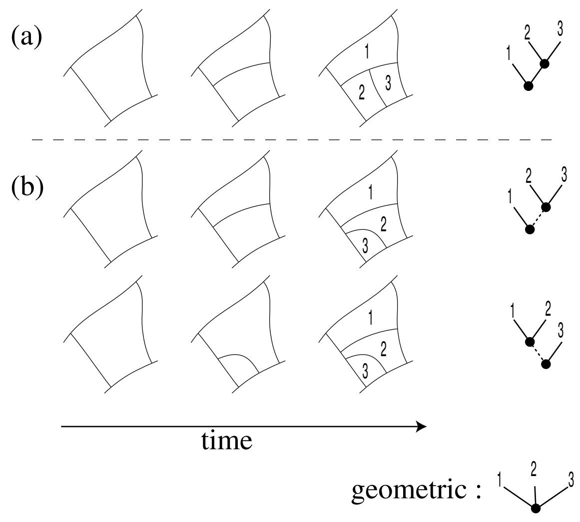

FIG. 5: (a) The generic case: the temporal and the geometrical trees are identical (drawn on the right). (b) Two different successions with distinct temporal trees (on the right) lead to the same final pattern and thus to the same geometrical tree (at the bottom). The geometrical tree can be obtained by fusing the intermediate domains in the temporal tree.

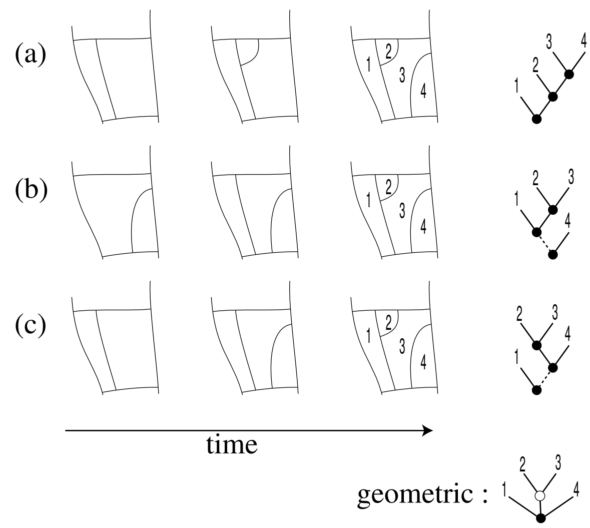

FIG. 6: Three possible successions leading to the same final pattern. Its geometrical tree is given at the the bottom, the different temporal trees on the right. The differences between case (b) and (c) are analog to the case (b) in fig. 5, the geometrical tree can be obtained by collapsing the points. In case (a), the intermediate domain indicated as an open circle in the geometrical tree is an artifact of the construction since it has never existed in the formation.

disk at the base of the genealogical tree. The first crack divides this mother domain into two daughter domains. They are of generation one and found on the next level of the tree. The major advantage of this approach is that we can now consider the two daughter domains independently, e.g. we can follow the different branches of the genealogical tree separately. For instance, the first generation domain found in box A is subdivided into two daughter domains. One of those is not divided any more and appears as such in the final pattern. It is represented by a number instead of a disk. The domains of the final pattern are labeled by numbers to allow the direct comparison with the geometrical tree.
In the temporal tree, a mother domain has exactly two daughter domains because the cracks are formed successively. The temporal tree of our example rises up to generation ten. The domain of the final pattern with the smallest generation is of generation two. The differences of the branch lengths are related to a dispersion of the domain sizes. All branches would have the same length and all final domains the same temporal generation. We shall emphasize that the intermediate domains, represented by the disks, are physical, meaning that they existed in a stage of the pattern formation.

The geometrical tree (fig. 4(b)) is built up in an analogous way, but is based on the geometrical order of the cracks. Two first order cracks divide the initial domain (generation zero) into three daughter domains (see also fig. 2). By contrast to the temporal tree, a mother domain can have more than two daughter domains. There are thus three branches connected to the corresponding representative disk. The three intermediate domains of geometrical generation one are subdivided into domains of generation two and so on. In the example under consideration, there are no domains with a geometrical generation higher than four. The fact that the number of geometrical generations is much lower than the number of temporal generations is mainly due to the fact that a mother domain often has more than two daughters in the geometrical tree.

B. Comparison between the temporal and the geometrical trees

For a more detailed comparison between the temporal and the geometrical trees, let us consider the branches in the boxes A, B and C of fig. 4 in the two trees. For simplicity, instead of dealing with the total experimental pattern, we sketched the corresponding generic configurations in fig. 5 and fig. 6.

The branches in box A in fig. 4 are identical in both trees. This situation corresponds to fig. 5(a). The case is different for the branches in B. They are similar, but we note that two intermediate domains in the temporal tree have collapsed to form one (arrow in fig. 4(a)). Figure 5(b) illustrates the geometrical reason. A domain is successively divided by two cracks into three domains. Since the second crack does not meet the first one, there is no geometrical indicator on the basis of the final pattern, that determines which one was first. From the geometrical point of view we have to consider the two cracks as equivalent; the initial domain is divided into three daughter domains as shown by the geometrical trees at the bottom of the figures. The two possible formation histories that lead to the same final configuration have different temporal trees, and furthermore, they are not physically equivalent. Since both cracks are formed in the same domain, the first one could have had an impact on the second. Note that the lack of information in the geometrical tree is due to the collapse of the two intermediate domains by the shrinking of the dashed segment in the temporal trees. Furthermore, the domains on the basis of the geometrical as well of the temporal tree correspond to the initial domain that is physical.

Let us now consider the branch C in fig. 4(a) and (b). They are different and, in contrast to the previous case, it is not possible to pass from the temporal tree to the geometrical tree by collapsing intermediate domains. The origin of the problem is explained in fig. 6. As in the case in fig. 5(b), the two cracks that divide the initial domain do not meet and there is no geometrical indication to distinguish their orders. In such a tricky case (a), this ambiguity leads to an error in the reconstruction of the domains. For this possible formation history, the intermediate domain represented by the open circle on the bottom of the figure has never existed and is an artifact of the construction. Nevertheless, we should consider this case as an exception.

C. Number of neighbors and number of sides

We should also discuss how the topology of the network evolves during the succession of domain divisions. In the context of foams, the topology is often represented by the dual graph (fig. 7). The vertexes of the dual graph represent the domains, the edges indicate the first neighborhood. A crack divides a domain, splits thus the corresponding vertex in the dual into two. Let us note n the number of neighbors of the mother domain. Two of its neighbors are neighbors of both daughter domains, while the other $n-2$ neighbors are distributed among the daughters. Taking into account that the sisters are neighbors, too, the number of neighbors of daughters (n_a,n_b) and mother are related by

$$n_a + n_b = n + 4 \tag{1}$$

We must not forget that the number of neighbors of the domains that are shared neighbors of the daughter domains also increase:

$$\begin{aligned} n_i &\rightarrow n_i + 1 \\ n_k &\rightarrow n_k + 1 \end{aligned} \tag{2}$$

The topology of spaced dividing pattern is often described in terms of the topological charges of the domains. The topological charge of a domains i with n_i

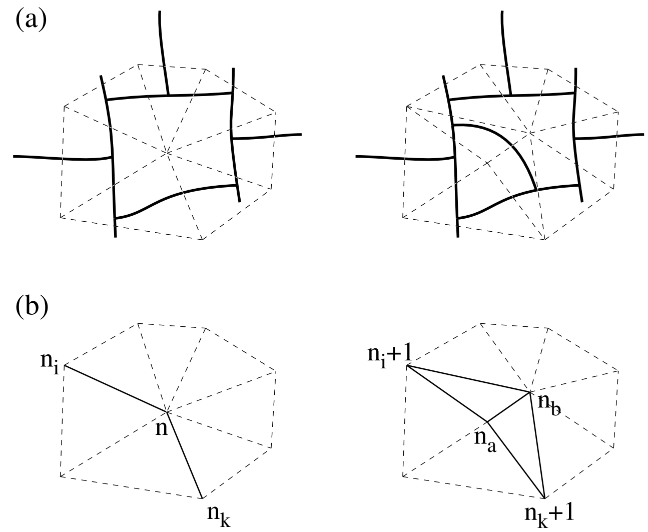

FIG. 7: Illustration how a new fracture affects the topology of the network. (a) The new crack is added. The dashed lines represent the dual graph. (b) The addition of the new crack in the dual graph. n is the number of neighbors of the mother domain, n_a and n_b the number of neighbors of the daughter domains. We must also account for the change of the number of neighbors of the adjacent domains n_i and n_k.

neighbors is defined as

$$q_{topo,i} = 6 - n_i \tag{3}$$

The equations 1 and 2 present thus the conservation of the total topological charge. The average number of neighbors in an extended pattern with N domains can be written as

$$\langle n \rangle = \frac{1}{N} \sum n_i = 6 - \frac{\sum q_{topo,i}}{N} \tag{4}$$

Since the total topological charge $\sum q_{topo,i}$ is conserved during the domain division, the average number of neighbors must converge to 6 by $1/N$. This is a particular demonstration of a very general result. The average number of neighbors in an extended network, hierarchical or not, must be six. It is a consequence of Euler's theorem on topology.

The numbers of neighbors are however not a good parameter to describe the structure of the crack network. We argued here-above that the domains after division become physically independent. The formation of a crack in one domain does however change the number of neighbors of the adjacent although independent domains. In the genealogical tree, these domains are found on different branches, and a meaningful parameter should display the corresponding independence.

The dynamics of the crack pattern is not governed by neighborhood relations, but by the shape of the domains. The shape of the domain defines the boundary conditions of the stress field in which the next fracture is nucleated

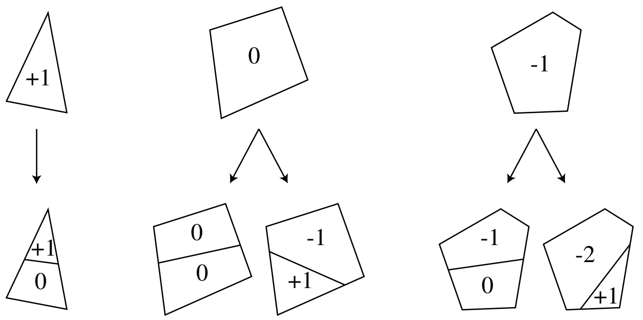

FIG. 8: The possible divisions of a triangle, a quadrangle and a pentagon. The numbers indicate the geometrical charges q_{geo} of the shapes.

and propagates. A simple parameter to describe the cell shape is its number of sides. We understand here a side as the part of the domain contour between two wedge-shaped singularities. A side can be curved, but its curvature is continuous. In particular, the 180° angles corresponding to fractures in neighboring domains do not present singularities in the cell shape. We do not account for them in considering the shape of the domains. The undivided domain on the left hand side of fig. 7 is four-sided while it has seven neighbors.

The concept of the successive domain division and the genealogical tree allows a direct understanding of the number of sides, which has to be four on average [19]. As shown in fig. 8, a four-sided domains can be divided either into two four-sided domains, or into a three and a five-sided domain. A triangular domains can be only divided into a quadrangle and a triangle. The number of sides of the 'daughter' domains s_a and s_b are in general related to the number of sides of the 'mother' domain s by

$$s_a + s_b = s + 4 \tag{5}$$

Introducing by analogy to the topological charge a *geometrical charge* by

$$q_{geo} = 4 - s \tag{6}$$

eq. 5 presents a conservation law in the genealogical tree:

$$q_{geo,a} + q_{geo,b} = q_{geo} \tag{7}$$

In contrast to the topological charge, the division of a domain does not affect the geometrical charges on the other branches of the genealogical tree. The average number of sides can be written in terms of the total geometrical charge $Q_{geo} = \sum q_{geo,i}$:

$$\langle s \rangle = \frac{1}{N} \sum s_i = 4 - \frac{Q_{geo}}{N} \tag{8}$$

In an extended pattern, it converges to four. The typical domain shape in the hierarchical crack pattern is therefore the quadrangle.

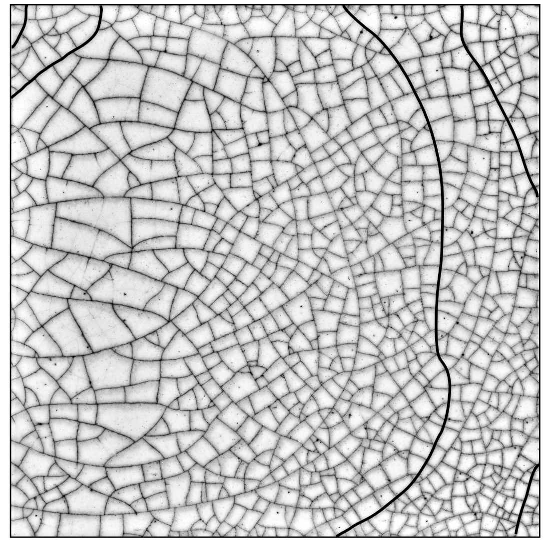

FIG. 9: A crack pattern in the glaze of an ceramic plate. The cracks cracks of first geometrical order are emphasized manually.

The conservation of the geometrical charge is one of possible conservation laws associated with genealogical tree. Another, trivial one is the conservation of the area: the sum of the areas of the daughter domains is equal to the area of the mother domain.

D. Application to an extended crack pattern

We above described an experiment using the drying of a latex gel because it lent itself to following the formation process. This experimental set-up is however limited because of the camera field. In order to consider a more extended pattern we analyzed the cracks in the glaze of a square ceramic plate shown in fig. 9. Since the formation of each crack emits a 'click' sound, its propagation velocity should therefore be close to the sound velocity of the material. Furthermore, the characteristic time between two cracks is in the order of seconds or minutes so that the condition of succession is clearly fulfilled. Since we are not able to follow the formation process (most cracks are formed in the cooling oven and are only visible later, after being colored with ink), we base our analysis on the geometrical tree which we constructed manually. In contrast to the gel experiment, the initial domain is given by the initial sample and not by the camera field.

The pattern is composed of 1620 undivided domains. Figure 10 shows the histogram of the domain generations. The open circles in the figure correspond to the undivided domains such as they appear in the final pattern. The distribution is quite large and irregular: most domains are of generations between 4 and 12. This is remarkable because if each domain in the formation process is divided into two approximately equal domains, one would expect that the final domains would be all of approximately the

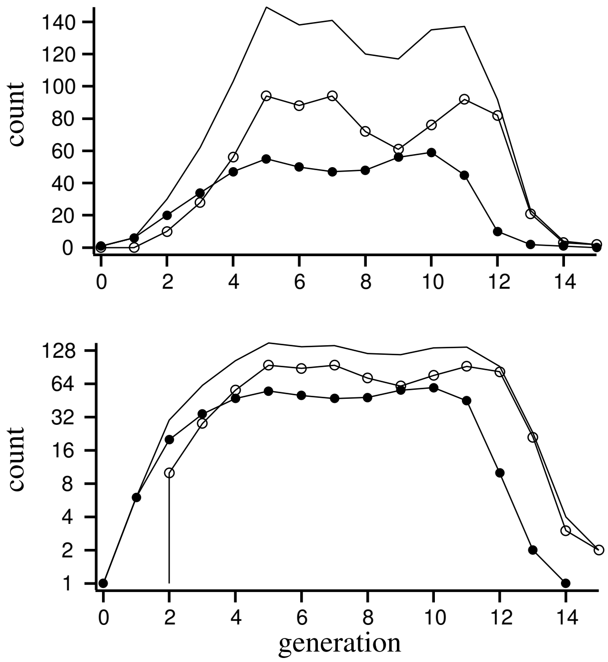

FIG. 10: The histograms of the domain generation (in linear and logarithmic scale). Open circles: undivided domains. Disks: divided domains. Line: all domains. The total number of domains is 1620.

same generation. The reason this is not the case can be understood by considering for instance the division of the first domain. We emphasized the fractures which divide it in fig. 9. Some of the daughter domains (domains of generation 1) are very small and are not much further divided. Let us also briefly consider all domains of the formation process, divided or not (continous line in the figure). In the temporal tree, each mother domain is divided into two daughter domains. Before reaching a cut-off due to the characteristic size of the final domains, one expects that the number of domains would increase like 2^g (g is the generation). However, in the geometrical tree, a mother domain is divided into two or more daughter domains. Assuming some regularity in the division process we would expect nevertheless an exponential increase like a^g with $a \gtrsim 2$. Such an exponential behavior is not observed in the data.

V. CONCLUSION

By contrast to soap-foams and similar space dividing patterns, there exists a crack morphology with an apparent hierarchical structure. We can consider the crack pattern (as is observed in the gaze of ceramics) as the model system for hierarchical space divisions. We introduced the concept of successive domain divisions and its representation by the genealogical trees as a framework for the comprehension of these patterns. By comparing the trees constructed on the basis of the formation process (temporal tree) and on the geometry of the final pattern (geometrical tree), we showed that it is (in the discussed limits) possible to reconstruct the relevant history by considering the finished pattern. This demonstrates that the formation process has its clear signature in the final geometry.
The formation process of the pattern is restrained by conservation laws associated with the genealogical tree. The conservation of the geometrical charge determines the average number of sides of the domains. It has to be four. This can be easily verified by counting the number of sides in an extended pattern.
Former studies [7, 8] in similar systems have shown that the final domains have a well defined characteristic size. A first analysis based on the proposed framework of an extended pattern in a ceramic plate revealed nevertheless that the generations of these final domains have a wide and irregular distribution; the genealogical tree have branches of very different lengths. They are the result of asymmetric domain divisions where one daughter is much larger than the other. As we will show in the second part of the present paper, this asymmetry is only observed at large scales and vanishes at small scales. We investigate therein the division of controlled domains. This study is directly based on the concept of successive domain divisions, which enables us to understand this complex pattern by studying of the divisions of single domains.

Acknowledgments

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