# Instabilité d'un nuage de gaz interstellaire

1* $\frac{\partial}{\partial t}(\rho_0 + \tilde{\rho}) + \mathrm{div}\left[(\rho_0+\tilde{\rho})(\vec{\tilde{u}} + \vec{u_0})\right] = 0$

$$\frac{\partial}{\partial t}\rho_0 + \rho_0 \,\mathrm{div}\,\vec{u_0} + \frac{\partial}{\partial t}\tilde{\rho} + \rho_0\,\mathrm{div}\,\vec{\tilde{u}} + \tilde{\rho}\,\mathrm{div}\,\vec{u_0} = 0$$

or $\mathrm{div}\,\vec{u_0} = 0$ $\Rightarrow$ $\frac{\partial}{\partial t}\tilde{\rho} + \rho_0\,\mathrm{div}\,\vec{\tilde{u}} + \tilde{\rho}\,\mathrm{div}\,\vec{u_0} = 0$

$\vec{u_0} = 0$

* $(\rho_0+\tilde{\rho})\left[\frac{\partial \vec{u_0}}{\partial t} + \frac{\partial \vec{\tilde{u}}}{\partial t} + (\vec{u_0}\cdot\vec{\nabla})(\vec{u_0}+\vec{\tilde{u}}) + (\vec{\tilde{u}}\cdot\vec{\nabla})(\vec{u_0}+\vec{\tilde{u}})\right]$

$$= -\vec{\nabla}p_0 - \vec{\nabla}\tilde{p} - \rho_0\vec{\nabla}(\phi_0+\phi) - \tilde{\rho}\vec{\nabla}(\phi_0+\tilde{\phi})$$

$$\rho_0\left[\frac{\partial \vec{u_0}}{\partial t} + \frac{\partial \vec{\tilde{u}}}{\partial t} + (\vec{u_0}\cdot\vec{\nabla})\vec{u_0} + (\vec{u_0}\cdot\vec{\nabla})\vec{\tilde{u}} + (\vec{\tilde{u}}\cdot\vec{\nabla})\vec{u_0} + (\vec{\tilde{u}}\cdot\vec{\nabla})\vec{\tilde{u}}\right] +$$

$$\tilde{\rho}\left[\frac{\partial \vec{u_0}}{\partial t} + (\vec{u_0}\cdot\vec{\nabla})\vec{u_0} + \frac{\partial \vec{\tilde{u}}}{\partial t} + (\vec{u_0}\cdot\vec{\nabla})\vec{\tilde{u}} + (\vec{\tilde{u}}\cdot\vec{\nabla})\vec{u_0} + (\vec{\tilde{u}}\cdot\vec{\nabla})\vec{\tilde{u}}\right]$$

$$= -\vec{\nabla}p_0 - \vec{\nabla}\tilde{p} - \rho_0\vec{\nabla}(\phi_0+\phi) - \tilde{\rho}\vec{\nabla}(\phi_0+\tilde{\phi})$$

d'où

$$\tilde{\rho}\left[\frac{\partial\vec{\tilde{u}}}{\partial t} + (\vec{u_0}\cdot\vec{\nabla})\vec{\tilde{u}} + (\vec{\tilde{u}}\cdot\vec{\nabla})\vec{\tilde{u}}\right] + \rho_0\left[\frac{\partial\vec{\tilde{u}}}{\partial t} + (\vec{u_0}\cdot\vec{\nabla})\vec{\tilde{u}} + (\vec{\tilde{u}}\cdot\vec{\nabla})\vec{\tilde{u}}\right]$$

$$= -\vec{\nabla}\tilde{p} - \rho_0\vec{\nabla}\phi - \tilde{\rho}\vec{\nabla}\phi_0 - \tilde{\rho}\vec{\nabla}\tilde{\phi}$$

* $\nabla^2\phi_0 + \nabla^2\tilde{\phi} = 4\pi G(\rho_0+\tilde{\rho}) \Leftrightarrow \nabla^2\tilde{\phi} = 4\pi G\tilde{\rho}$
* $p_0 + \tilde{p} = c_s^2(\rho_0+\tilde{\rho}) \Leftrightarrow \tilde{p} = c_s^2\tilde{\rho}$

## 2- linearisation

* $\frac{\partial}{\partial t}\tilde{\rho} + \rho_0\,\mathrm{div}\,\vec{\tilde{u}} = 0$ (1)
* $\rho_0\frac{\partial\vec{\tilde{u}}}{\partial t} = -\vec{\nabla}\tilde{p} - \rho_0\vec{\nabla}\tilde{\phi}$ (2)
* $\vec{\nabla}^2\tilde{\phi} = 4\pi G\tilde{\rho}$ (3)
* $\tilde{p} = c_s^2\tilde{\rho}$ (4)

3 - on prend la divergence de l'équation (2)

$$\mathrm{div}\left[\rho_0 \frac{\partial}{\partial t}\vec{\tilde{u}}\right] = -\,\mathrm{div}\,\vec{\mathrm{grad}}\,\tilde{p} \mp \rho_0\,\mathrm{div}\,\vec{\mathrm{grad}}\,\tilde{\phi}$$

or $\frac{\partial}{\partial t}\rho_0 = 0 \Rightarrow \mathrm{div}\,\rho_0 \frac{\partial}{\partial t}\vec{\tilde{u}} = \frac{\partial}{\partial t}\mathrm{div}\left[\rho_0 \vec{\tilde{u}}\right]$

or $\mathrm{div}\,\rho_0 \vec{\tilde{u}} = -\frac{\partial}{\partial t}\tilde{\rho}$ eq (1)

$\left.\right\}\ \mathrm{div}\,\rho_0 \frac{\partial}{\partial t}\vec{\tilde{u}} = -\frac{\partial^2}{\partial t^2}\tilde{\rho}$

d'où

$$-\frac{\partial^2}{\partial t^2}\tilde{\rho} + \nabla^2 \tilde{p} + \rho_0 \nabla^2 \tilde{\phi} = 0$$

enfin $\nabla^2 \tilde{\phi} = 4\pi G \tilde{\rho}$ et $\tilde{\rho} = \frac{1}{c_s^2}\tilde{p}$ on a

$$-\frac{1}{c_s^2}\frac{\partial^2}{\partial t^2}\tilde{p} + \nabla^2 \tilde{p} + \rho_0 \frac{4\pi G}{c_s^2}\tilde{p} = 0$$

$$\boxed{-\frac{\partial^2}{\partial t^2}\tilde{p} + c_s^2 \nabla^2 \tilde{p} + \rho_0 4\pi G \tilde{p} = 0}$$

$\tilde{p} = p\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$ $\frac{\partial^2}{\partial t^2}\tilde{p} = \mp \omega^2 \tilde{p}$ $\nabla^2 \tilde{p} = -k^2 \tilde{p}$

soit $+\omega^2 p - c_s^2 k^2 p + \rho_0 4\pi G p = 0$

4 - $\boxed{+\omega^2 = c_s^2 k^2 - 4\pi \rho_0 G}$

5 - $\cancel{k^2 = \frac{1}{c_s^2}(4\pi \rho_0 G + \omega^2) \Rightarrow k = \pm}$

$\omega^2 = c_s^2 k^2 - 4\pi \rho_0 G$ $\omega = \pm\sqrt{c_s^2 k^2 - 4\pi \rho_0 G}$

d'où $\boxed{k_c^2 = \frac{4\pi \rho_0 G}{c_s^2}}$

Examen de turbulence du 3 avril 2017 

## II Sillage turbulent d'1 objet propulsé

1) la contrainte visqueuse, $\sigma'_{xy} = \eta \frac{\partial \bar{u}}{\partial y}$, s'annule à chaque extremum de $\bar{u}$ en $y=0$ et en $y = \pm y_{min(\bar{u})}$.

la contrainte turbulente, $\tau_{xy} = -\rho \overline{u'v'}$, change de signe en $y=0$ par symétrie.

2) $$\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} - \frac{\partial \overline{u'^2}}{\partial x} - \frac{\partial \overline{u'v'}}{\partial y} + \nu\left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2}\right)$$

$$\frac{\partial \bar{v}}{\partial t} + \bar{u}\frac{\partial \bar{v}}{\partial x} + \bar{v}\frac{\partial \bar{v}}{\partial y} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial y} - \frac{\partial \overline{u'v'}}{\partial x} - \frac{\partial \overline{v'^2}}{\partial y} + \nu\left(\frac{\partial^2 \bar{v}}{\partial x^2} + \frac{\partial^2 \bar{v}}{\partial y^2}\right)$$

3) L'éq. de Reynolds projetée selon y se réduit à $\frac{\partial \bar{p}}{\partial x} = -\rho\frac{\partial \overline{v'^2}}{\partial y}$

ssi
- écoulem^t statistiq^t stationnaire $\frac{\partial \bar{v}}{\partial t} = 0$
- viscosité negligeable $\nu = 0$.
- $\left\|\frac{\partial \overline{u'v'}}{\partial x}\right\| \ll \left\|\frac{\partial \overline{v'^2}}{\partial y}\right\|$ car $\ell(x) \ll x$.
- $\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \Rightarrow \bar{v} \simeq \frac{\ell(x)}{x} U_s$.
- $O.g.\left(\bar{u}\frac{\partial \bar{v}}{\partial x}\right) = U_0 U_s \frac{\ell(x)}{x^2} \gg O.g.\left(\bar{v}\frac{\partial \bar{v}}{\partial y}\right) = \frac{U_s^2 \ell(x)}{x}$
- $O.g.\left(\bar{u}\frac{\partial \bar{v}}{\partial x}\right) = U_0 U_s \frac{\ell(x)}{x^2} \ll O.g.\left(\frac{\partial \overline{v'^2}}{\partial y}\right) = \frac{u^{*2}}{\ell(x)}$

ssi $\boxed{\left(\frac{\ell}{x}\right)^2 \frac{U_0 U_s}{u^{*2}} \ll 1}$

Ds ce cas $\frac{\partial \bar{p}}{\partial x} = -\rho \frac{\partial \overline{v'^2}}{\partial y}$ et, en intégrant, on trouve que

$$\bar{p}(y) = P_0 - \rho\overline{v'^2}(y) + \underbrace{f(x)}_{=0 \text{ car } \bar{p} = P_0 \text{ à l'}\infty}$$

4) L'éq. de Reynolds selon $x$ :

- $$-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} = \frac{\partial \overline{v'^2}}{\partial x} \ll \frac{\partial \overline{u'v'}}{\partial y}.$$

- $$\text{O.g.}: \left(\bar{u}\frac{\partial \bar{u}}{\partial x}\right) = \frac{U_0 U_s}{x} \gg \text{O.g.}\left(\bar{v}\frac{\partial \bar{u}}{\partial y}\right) = \frac{U_s^2}{x}$$

- $$\bar{u}\frac{\partial \bar{u}}{\partial x} = (U_0 - \Delta U)\frac{\partial (U_0 - \Delta U)}{\partial x} = -(U_0 - \Delta U)\frac{\partial \Delta U}{\partial x} \simeq -U_0 \frac{\partial \Delta U}{\partial x}.$$

Finalemᵗ, on obtient : $\boxed{U_0 \frac{\partial \Delta U}{\partial x} = \frac{\partial \overline{u'v'}}{\partial y}}$.

5) $$\int_{-\infty}^{+\infty} U_0\, y^2 \frac{\partial \Delta U}{\partial x}\, dy = -\int_{-\infty}^{+\infty} \nu_T(x)\, y^2 \frac{\partial^2 \bar{u}}{\partial y^2}\, dy$$

$$U_0 \frac{\partial}{\partial x}\int_{-\infty}^{+\infty} y^2 \Delta U\, dy = \nu_T(x)\int_{-\infty}^{+\infty} y^2 \frac{\partial^2 \Delta U}{\partial y^2}\, dy$$

on intègre par parties : $u = y^2$ et $v' = \frac{\partial^2 \Delta U}{\partial y^2}$

$$= \nu_T(x)\left\{\underbrace{\left[y^2 \frac{\partial \Delta U}{\partial y}\right]_{-\infty}^{+\infty}}_{0} - \int_{-\infty}^{+\infty} 2y \frac{\partial \Delta U}{\partial y}\, dy\right\}$$

puis $u = y$ et $v' = \frac{\partial \Delta U}{\partial y}$

$$= -2\nu_T(x)\left\{\underbrace{\left[y\, \Delta U\right]_{-\infty}^{+\infty}}_{0} - \int_{-\infty}^{+\infty} \Delta U\, dy\right\} = 0$$

$\Rightarrow I = \int_{-\infty}^{+\infty} y^2 \Delta U(x,y)\, dy$ est indpdt de $x$.

6) $\Delta U(x,y) = U_s(x)\, f(\xi)$ où $f$ est paire et $\begin{cases} f(0) = 1 \\ f(\pm 1) = 0 \text{ et } f(\pm\infty) = 0 \end{cases}$

$$I = \int_{-\infty}^{+\infty} y^2 \Delta U\, dy = \ell^3(x)\, U_s(x) \int_{-\infty}^{+\infty} \xi^2 f(\xi)\, d\xi$$

soit $\ell^3(x)\, U_s(x) = J = \frac{I}{\int_{-\infty}^{+\infty} \xi^2 f(\xi)\, d\xi}$

$\Rightarrow 3\ell^2(x)\, U_s(x) \dfrac{d\ell}{dx} + \ell^3 \dfrac{dU_s}{dx} = 0 \quad \Rightarrow \quad \boxed{\dfrac{dU_s}{dx} = -3 \dfrac{U_s(x)}{\ell(x)} \dfrac{d\ell}{dx}}$

7) $\nu_T(x) = \nu\, \ell(x)\, U_s(x)$ est le produit de la taille des structures turbulentes par leur vit. caractéristique.

8) $\dfrac{\partial}{\partial y} = \dfrac{1}{\ell(x)} \dfrac{\partial}{\partial \xi}$ et $\dfrac{\partial}{\partial x} = -\dfrac{1}{\ell(x)} \dfrac{d\ell}{dx} \xi \dfrac{\partial}{\partial \xi}$

$$U_0 \frac{dU_s}{dx} f(\xi) - \frac{U_0 U_s}{\ell} \frac{d\ell}{dx} \xi f' = \nu \frac{U_s^2}{\ell} f''$$

on obtient finalem$^t$ : $\boxed{\dfrac{-f''}{3f + \xi f'} = -\dfrac{U_0}{\nu} \dfrac{1}{U_s(x)} \dfrac{d\ell}{dx} = \alpha}$ (*)

9) $\dfrac{1}{U_s(x)} \dfrac{d\ell}{dx} = \alpha \Rightarrow \boxed{m = n+1}$

$\ell^3(x)\, U_s(x) = c^{te} \Rightarrow \boxed{n = -3m}$

$\Rightarrow \boxed{\begin{array}{l} m = 1/4 \\ n = -3/4 \end{array}}$ $\quad U_s(x) \propto x^{-3/4}$, $\ell(x) \propto x^{1/4}$

$Re_T(x) = \dfrac{U_s(x)\,\ell(x)}{\nu} \propto x^{-1/2}$ $\Rightarrow$ retour au laminaire loin derrière l'objet, contrairement au sillage d'1 objet passif.

$\nu_T(x) \propto x^{-1/2}$ $\Rightarrow$ la contrainte turbulente disparaît loin de l'objet $\Rightarrow$ retour au laminaire.

10) On s'attend à ce que $\nu_T = 0$ loin de l'axe $y \gg \ell(x)$.
Il aurait fallu tenir compte d'1 décroissance de $\nu_T$ avec $y$ pour que notre modèle soit + pertinent.
Ce modèle est adapté proche du centre en $y \simeq 0$ et marche moins bien loin du centre $y \gg \ell(x)$.

11) $f'' + 3f + \xi f' = 0$ d'après (*) si $\alpha = 1$

$$f(\xi) = \frac{\partial^2}{\partial \xi^2}\left[\exp\left(-\tfrac{1}{2}\xi^2\right)\right] = (\xi^2 - 1)\exp\left(-\tfrac{1}{2}\xi^2\right)$$

$$f'(\xi) = (3\xi - \xi^3)\exp\left(-\tfrac{1}{2}\xi^2\right)$$
$$f''(\xi) = (3 - 6\xi^2 + \xi^4)\exp\left(-\tfrac{1}{2}\xi^2\right)$$

on injecte ds $f'' + 3f + \xi f' = 0$ et ça marche. $f(\xi)$ est bien solution.

on vérifie que $f(\pm\infty) = f(\pm 1) = 0$

$f(0) = -1 \Rightarrow \bar{u}(x,0) = U_0 - \Delta U(x,0)$
$= U_0 - U_s(x) f(0) = U_0 + U_s$.

12) $-\overline{u'v'} = g(\xi)\, U_s^2(x) = \nu_T(x) \dfrac{\partial \bar{u}}{\partial y}$

$$\nu\, \cancel{\ell(x)}\, U_s\cancel{(x)} \frac{1}{\cancel{\ell(x)}} U_s\cancel{(x)} f' = U_s^{\cancel{2}}\cancel{(x)}\, g$$

$\Rightarrow \boxed{g = \nu f'}$ $\quad g$ est impaire

et $g(\xi) = (3\xi - \xi^3)\exp\left(-\tfrac{1}{2}\xi^2\right)$.