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Dispersion de polluants dans l'atmosphère

1) $\dfrac{\partial u}{\partial x} \sim \dfrac{\partial w}{\partial z}$

$$\frac{U}{\ell(z)} \sim \frac{W}{L} \Rightarrow \boxed{U \sim \frac{\ell(z)}{L} W} \ll W.$$

2) a) $$\overline{\frac{\partial(\bar{c}+c')}{\partial t}} + \overline{(\bar{u}_i + u'_i)\frac{\partial(\bar{c}+c')}{\partial x_i}} = D\,\overline{\frac{\partial^2(\bar{c}+c')}{\partial x_i^2}}$$

$\dfrac{\partial}{\partial t} \to 0$: stat.

$$\bar{u}_i \frac{\partial \bar{c}}{\partial x_i} + \underbrace{\overline{\bar{u}_i \frac{\partial c'}{\partial x_i}}}_{0} + \underbrace{\overline{u'_i \frac{\partial \bar{c}}{\partial x_i}}}_{0} + \overline{u'_i \frac{\partial c'}{\partial x_i}} = D\frac{\partial^2 \bar{c}}{\partial x_i^2} + \underbrace{D\,\overline{\frac{\partial^2 c'}{\partial x_i^2}}}_{0}$$

et : $$\frac{\partial \overline{u'_i c'}}{\partial x_i} = \overline{u'_i \frac{\partial c'}{\partial x_i}} + \underbrace{\overline{c' \frac{\partial u'_i}{\partial x_i}}}_{0 \text{ par incompressibilité}}$$

Finalement :

$$\boxed{\bar{u}_i \frac{\partial \bar{c}}{\partial x_i} + \frac{\partial}{\partial x_i}\overline{u'_i c'} = D\frac{\partial^2 \bar{c}}{\partial x_i^2}}$$

(b) $$\bar{u}_i \frac{\partial \bar{c}}{\partial x_i} + \frac{\partial}{\partial x_i}\overline{u'_i c'} = \mathcal{D}\frac{\partial^2 \bar{c}}{\partial x_i^2}$$

$$\underbrace{\bar{u}\frac{\partial \bar{c}}{\partial x}}_{U\frac{C}{\ell}} + \underbrace{\bar{w}\frac{\partial \bar{c}}{\partial z}}_{\frac{WC}{L} = \frac{\cancel{K}}{\ell}U\frac{C}{\cancel{K}}} + \underbrace{\frac{\partial}{\partial x}\overline{u'c'}}_{\frac{1}{\ell}u^*c^*} + \underbrace{\frac{\partial}{\partial z}\overline{w'c'}}_{\frac{u^*c^*}{L}} = \underbrace{\mathcal{D}\frac{\partial^2 \bar{c}}{\partial x^2}}_{\frac{\mathcal{D}C}{\ell^2}} + \underbrace{\mathcal{D}\frac{\partial^2 \bar{c}}{\partial z^2}}_{\frac{\mathcal{D}C}{L^2}}$$

Diffusion $\ll$ transport turbulent ssi :

$$\frac{\mathcal{D}C}{\ell^{\cancel{2}}} \ll \frac{u^*c^*}{\cancel{\ell}}$$

$$\frac{C}{c^*} \ll \frac{u^*\ell}{\mathcal{D}} = Pe$$

D'après la photographie $Pe \gg 1$.

(c) On obtient donc

$$\boxed{\bar{u}\frac{\partial \bar{c}}{\partial x} + \bar{w}\frac{\partial \bar{c}}{\partial z} + \frac{\partial}{\partial x}\overline{u'c'} = 0} \qquad \text{(eq. 3)}$$

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3) $\overline{w} = W_0(z)\, f(\xi)$

$\overline{c} = C_0(z)\, g(\xi)$

$+\overline{u'c'} = W_0(z)\, C_0(z)\, h(\xi)$

avec $\xi = \dfrac{x}{\ell(z)}$

$\overline{w}(\ell, z) = \dfrac{1}{2}\overline{w}(0,z)$

f est 1 f° paire avec $f(0) = 1$ et $f(1) = \dfrac{1}{2}$.

ⓐ $g(\xi)$ et $h(\xi)$ | tendent vers 0 en $\xi \to \pm\infty$
| sont max au centre.

ⓑ $\dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial \xi}\dfrac{\partial \xi}{\partial x} = \dfrac{1}{\ell}\dfrac{\partial}{\partial \xi}$

$\dfrac{\partial}{\partial z} = \dfrac{\partial}{\partial \xi}\dfrac{\partial \xi}{\partial z} = \dfrac{\partial}{\partial \xi}\dfrac{\partial \xi}{\partial \ell}\dfrac{d\ell}{dz} = -\dfrac{x}{\ell^2}\dfrac{d\ell}{dz}\dfrac{\partial}{\partial \xi}$

$\rightarrow \dfrac{\partial}{\partial z} = -\dfrac{1}{\ell}\dfrac{d\ell}{dz}\,\xi\,\dfrac{\partial}{\partial \xi}$

ⓒ $\dfrac{\partial \overline{u}}{\partial x} + \dfrac{\partial \overline{w}}{\partial z} = 0$

$\dfrac{1}{\ell}\dfrac{\partial}{\partial \xi}\left(\overline{u}\right) + \dfrac{\partial}{\partial z}\left(W_0(z)\, f(\xi)\right) = 0$

$$\frac{1}{\ell}\frac{\partial \bar{u}}{\partial \zeta} = -\frac{dW_0}{dz} f(\zeta) + \frac{W_0(z)}{\ell}\frac{d\ell}{dz}\zeta\frac{\partial f}{\partial \zeta}$$

$$\frac{\partial \bar{u}}{\partial \zeta} = -\ell\frac{dW_0}{dz} f + W_0\frac{d\ell}{dz}\zeta\frac{\partial f}{\partial \zeta}$$

$$\boxed{\bar{u} = -\ell\int_0^\zeta\left(\frac{dW_0}{dz}f - \frac{W_0}{\ell}\frac{d\ell}{dz}\zeta f'\right)d\zeta}$$

d) $$-\ell\int_0^\zeta\left(\frac{dW_0}{dz}f - \frac{W_0}{\ell}\frac{d\ell}{dz}\zeta f'\right)d\zeta \cdot \frac{1}{\ell}\frac{\partial}{\partial\zeta}(C_0 g)$$

$$+ W_0 f g\frac{dC_0}{dz} + W_0 f\left(-\frac{1}{\ell}\frac{d\ell}{dz}\zeta\frac{\partial}{\partial\zeta}\right)(C_0 g)$$

$$+\frac{1}{\ell}\frac{\partial}{\partial\zeta}(W_0 C_0 h) = 0$$

$$-\frac{\cancel{\ell} C_0}{\cancel{\ell}}\frac{dW_0}{dz}g'\int_0^\zeta f\,d\zeta + \frac{W_0}{\ell}\frac{d\ell}{dz}C_0 g'\int_0^\zeta \zeta f'\,d\zeta + W_0\frac{dC_0}{dz}fg$$

$$-\frac{W_0 C_0}{\ell}\frac{d\ell}{dz}\zeta f g' + \frac{W_0 C_0}{\ell}h' = 0$$

On multiplie par $\ell / W_0 C_0$.

$$\boxed{-\frac{\ell}{W_0}\frac{dW_0}{dz}g'\int_0^\zeta f\,d\zeta + \frac{d\ell}{dz}g'\int_0^\zeta \zeta f'\,d\zeta + \frac{\ell}{C_0}\frac{dC_0}{dz}fg - \frac{d\ell}{dz}\zeta f g' + h' = 0}$$

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4) a) $\frac{dl}{dz} = \text{cste}$ D'où $\boxed{l \propto z}$

b) $$\frac{\partial \bar{u}\bar{w}}{\partial x} + \frac{\partial \bar{w}^2}{\partial z} + \frac{\partial}{\partial x}\overline{u'w'} = 0 \qquad (5)$$

On intègre selon x

$$\int_{-\infty}^{+\infty} \frac{\partial \bar{u}\bar{w}}{\partial x}\,dx + \int_{-\infty}^{+\infty} \frac{\partial \bar{w}^2}{\partial z}\,dx + \int_{-\infty}^{+\infty} \frac{\partial \overline{u'w'}}{\partial x}\,dx = 0$$

$$\frac{\partial}{\partial z}\int_{-\infty}^{+\infty} \bar{w}^2\,dx = -\left[\bar{u}\bar{w}\right]_{-\infty}^{+\infty} - \left[\overline{u'w'}\right]_{-\infty}^{+\infty} = 0$$

D'où $\boxed{\phi = \int_{-\infty}^{+\infty} \rho\,\bar{w}^2\,dx = \text{cste}}$ indpdte de z.

$$\int_{-\infty}^{+\infty} \bar{w}^2\,dx = \text{cste} \qquad \xi = \frac{x}{l(z)} \qquad x = l\xi \qquad dx = l\,d\xi$$

$$\int_{-\infty}^{+\infty} W_0^2(z)\, f^2(\xi)\, l\,d\xi = \text{cste}$$

$$l(z)\, W_0^2(z) \underbrace{\int_{-\infty}^{+\infty} f^2(\xi)\,d\xi}_{\simeq O(1)} = \text{cste}$$

$$l(z)\, W_0^2(z) \sim \text{cste} \qquad W_0^2(z) \sim \frac{1}{l(z)} \sim \frac{1}{z}$$

$$\boxed{W_0 \propto z^{-1/2}}$$

c) $$\frac{\partial \bar{u}\bar{c}}{\partial x} + \frac{\partial \bar{w}\bar{c}}{\partial z} + \frac{\partial \overline{u'c'}}{\partial x} = 0$$

on intègre selon x :

$$\int_{-\infty}^{+\infty} \frac{\partial \bar{u}\bar{c}}{\partial x}\, dx + \int_{-\infty}^{+\infty} \frac{\partial \bar{w}\bar{c}}{\partial z}\, dx + \int_{-\infty}^{+\infty} \frac{\partial \overline{u'c'}}{\partial x}\, dx = 0$$

$$\frac{\partial}{\partial z}\int_{-\infty}^{+\infty} \bar{w}\bar{c}\, dx = -\underbrace{\cancel{\left[\bar{u}\bar{c}\right]_{-\infty}^{+\infty}}}_{0} - \underbrace{\cancel{\left[\overline{u'c'}\right]_{-\infty}^{+\infty}}}_{0} = 0$$

$$\phi_c = \int_{-\infty}^{+\infty} \bar{w}\bar{c}\, dx = \text{cste} \quad \text{indpdte de } z.$$

$$\int_{-\infty}^{+\infty} W_0(z)\, C_0(z)\, f\, g\, \ell(z)\, d\zeta = \text{cste}$$

$$\ell(z)\, W_0(z)\, C_0(z) \underbrace{\int_{-\infty}^{+\infty} f\, g\, d\zeta}_{\sim O(1)} = \text{cste}$$

D'où $\quad \ell(z)\, W_0(z)\, C_0(z) \sim \text{cste}$

$$C_0(z) \sim \frac{1}{\ell(z) W_0(z)}$$

$$C_0(z) \propto \frac{1}{z \cdot z^{-1/2}} \longrightarrow \boxed{C_0(z) \propto z^{-1/2}}$$

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5) $\tau_{xz} = \rho \nu_T \dfrac{\partial \overline{w}}{\partial x}$

a) Par analogie avec le modèle de viscosité turbulente

Modèle de diffusivité turbulente

$$\boxed{-\overline{u'c'} = D_T \frac{\partial \overline{c}}{\partial x}}$$

b) $\overline{u'c'} = W_0(z)\, C_0(z)\, h(\xi) = -D_T \dfrac{\partial \overline{c}}{\partial x}$

$$= -D_T \cdot \frac{1}{\ell} \frac{\partial}{\partial \xi}\left(C_0(z)\, g(\xi)\right)$$

$$W_0(z)\, C_0\cancel{(z)}\, h(\xi) = -\frac{D_T}{\ell} C_0\cancel{(z)}\, g'(\xi)$$

$$\boxed{h(\xi) = -\frac{D_T}{\ell W_0} g'(\xi)}$$

c) $-\dfrac{\ell}{W_0}\dfrac{dW_0}{dz} g' \displaystyle\int_0^\xi f\, d\xi + \dfrac{d\ell}{dz} g' \underbrace{\int_0^\xi \xi f'\, d\xi} + \dfrac{\ell}{C_0}\dfrac{dC_0}{dz} f g$

$$-\frac{d\ell}{dz}\xi f g' = \frac{D_T}{\ell W_0} g''$$

$$\int_0^\xi \xi f'\, d\xi = \xi f - \int_0^\xi f\, d\xi$$

$$-\frac{\ell}{W_0}\frac{dW_0}{dz} g' \int_0^\xi f\, d\xi + \cancel{\frac{d\ell}{dz}\xi f g'} - \frac{d\ell}{dz} g' \int_0^\xi f\, d\xi + \frac{\ell}{C_0}\frac{dC_0}{dz} f g$$

$$\cancel{-\frac{d\ell}{dz}\xi f g'} = \frac{D_T}{\ell W_0} g''$$

On obtient :

$$\frac{\ell}{C_0}\frac{dC_0}{dz} f g - \left(\frac{\ell}{W_0}\frac{dW_0}{dz} + \frac{d\ell}{dz}\right) g' \int_0^{\xi} f \, d\xi = \frac{D_T}{\ell W_0} g''$$

On utilise $\ell W_0 C_0 = \text{cte}$

$$\ell W_0 \frac{dC_0}{dz} + \ell C_0 \frac{dW_0}{dz} + C_0 W_0 \frac{d\ell}{dz} = 0$$

D'où $$\frac{dC_0}{dz} = -\frac{C_0}{W_0}\frac{dW_0}{dz} - \frac{C_0}{\ell}\frac{d\ell}{dz}$$

on remplace :

$$\frac{\ell}{\cancel{C_0}}\left(-\frac{\cancel{C_0}}{W_0}\frac{dW_0}{dz}\right) f g + \frac{\ell}{\cancel{C_0}}\left(-\frac{\cancel{C_0}}{\cancel{\ell}}\frac{d\ell}{dz}\right) f g - \left(\frac{\ell}{W_0}\frac{dW_0}{dz} + \frac{d\ell}{dz}\right) g' \int_0^{\xi} f \, d\xi$$

$$= \frac{D_T}{\ell W_0} g''$$

$$-\left(\frac{d\ell}{dz} + \frac{\ell}{W_0}\frac{dW_0}{dz}\right) \underbrace{\left\{ f g + g' \int_0^{\xi} f \, d\xi \right\}}_{= \frac{\partial}{\partial \xi}\left(g \int_0^{\xi} f \, d\xi\right)} = \frac{D_T}{\ell W_0} g''$$

Finalement :

$$\left(\frac{d\ell}{dz} + \frac{\ell}{W_0}\frac{dW_0}{dz}\right) \frac{\partial}{\partial \xi}\left(g \int_0^{\xi} f \, d\xi\right) = -\frac{D_T}{\ell W_0} g''$$

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On intègre / ξ.

$$\boxed{\left(\frac{dl}{dz} + \frac{l}{W_0}\frac{dW_0}{dz}\right) g \int_0^{\xi} f \, d\xi = -\frac{D_T}{l W_0} g'}$$

d

$$\underbrace{-\frac{g'}{g\int_0^{\xi} f \, d\xi}}_{f^\circ \text{ de } \xi} = \underbrace{\frac{l W_0}{D_T}\left(\frac{dl}{dz} + \frac{l}{W_0}\frac{dW_0}{dz}\right)}_{f^\circ \text{ de } z} = \text{cste}.$$

$$\frac{g'}{g} = -\int_0^{\xi} f \, d\xi$$

d'où $\boxed{g(\xi) = +B \exp\left(-\int_0^{\xi} F(\xi)\, d\xi\right)}$

si $f(\xi) \simeq 1$

$$F(\xi) = \int_0^{\xi} \overset{1}{\overset{\shortparallel}{f(\xi)}} \, d\xi \sim \xi$$

$$\int_0^{\xi} F(\xi)\, d\xi \simeq \int_0^{\xi} \xi \, d\xi = \frac{\xi^2}{2}$$

D'où $\boxed{g(\xi) \simeq B \exp\left(-\frac{\xi^2}{2}\right)}$

l'approxima° $f(\xi) \simeq 1$ est valide au centre du jet.