Counter-rotation in an orbitally shaken glass of beer: Supplementary Material

We present a theoretical model that describes the motion of a floating circular raft in the orbital sloshing problem. In section **A**, we specify the orbital sloshing flow with free surface. In **B**, we derive equations for the motion of the raft. In sections **C** and **D**, we calculate first and second order approximations of the motion of the raft to describe its gyration and its counter-rotation. In section **E**, we explain the counter-rotation of the raft as a result of the interaction with the boundary layer.

A. Flows in an orbitally shaken cylinder. – A cylinder of radius R filled with fluid up to height H is being orbitally displaced as $\mathbf{r}_c = A(\cos\Omega t\,\mathbf{e}_x + \cos\Omega t\,\mathbf{e}_y)$. Here A is the amplitude of the displacement and Ω its frequency. This orbital translation drives a flow $\mathbf{u}(\mathbf{r},\mathbf{t})$ that we describe using cylindrical coordinates (r,θ,z) in the moving frame of reference attached to the cylinder.

Potential flow theory provides a linear and inviscid approximation of the fluid flow [3,4]. For forcing frequencies Ω that are lower than the natural frequencies of the gravity waves, we have

$$\boldsymbol{u}(\mathbf{r},t) = \nabla\phi \quad \text{with} \quad \phi = \Omega R^2\,\frac{2\chi}{k_1^2-1}\,\frac{J_1(k_1 r/R)}{J_1(k_1)}\,\frac{\cosh(k_1(z+H)/R)}{\cosh(k_1 H/R)}\,\sin(\theta-\Omega t). \tag{1}$$

Here $k_1 \simeq 1.841$ and

$$\chi = \frac{\epsilon}{(\omega_1/\Omega)^2-1}\,, \tag{2}$$

with $\epsilon = A/R$ and $\omega_1 = \sqrt{g k_1 \tanh(k_1 H/R)}$ the gravity wave eigenfrequency. The surface reaches a height $z=\eta$ with

$$\eta = R\,\frac{2\chi k_1}{k_1^2-1}\,\frac{J_1(k_1 r/R)}{J_1(k_1)}\,\tanh(k_1 H/R)\cos(\theta-\Omega t). \tag{3}$$

This solution only includes the dominant wave. The full solution is given in [3,4].

Near the boundaries of the cylinder the inviscid potential flow model needs to be corrected in order to satisfy the no-slip boundary condition. We introduce an exponential boundary layer correction to the flow so that

$$\mathbf{u}(\mathbf{r},t) = \nabla\phi - \nabla\phi|_{r=R}\,e^{-(R-r)/\delta}\ . \tag{4}$$

The boundary layer has thickness $\delta \ll R$ and δ will be a tunable parameter. This boundary layer is a very crude approximation of the real boundary layer near the contact line, but it is adequate to capture the essential physics that explains the counter-rotation.

Nonlinearities in the bulk and in the boundary layers create a weak $O(\chi^2)$ correction to the flow under the wave. A second order, more precise model of the flow in the bulk is

$$\mathbf{u}(\mathbf{r},t) = \nabla\phi(\mathbf{r},t) + \overline{\mathbf{u}}(\mathbf{r}) + \mathbf{u}'(\mathbf{r},t). \tag{5}$$

Next to the oscillatory potential wave, we find the steady streaming flow $\overline{\mathbf{u}}(\mathbf{r})$ as the Eulerian mean flow and some time dependent harmonics $\mathbf{u}'(\mathbf{r},t)$. The steady streaming flow was measured in Ref. [4], but no analytical expression is available.

B. Equations of motion for the raft. – We consider the motion of a set of N identical particles of mass m submerged in a fluid moving at speed $\mathbf{u}(\mathbf{r},t)$. The position $\mathbf{r}_i(t)$ and speed $\mathbf{v}_i(t) = d\mathbf{r}_i/dt$ of a particle i satisfy a fundamental force balance

$$\underbrace{m\left(\frac{d^2\mathbf{r}_i}{dt^2}+\frac{d^2\mathbf{r}_c}{dt^2}\right)}_{inertia} = \underbrace{\alpha_i\left(\mathbf{u}(\mathbf{r}_i,t)-\mathbf{v}_i\right)}_{drag\ force} + \underbrace{\sum_{j\neq i}\mathbf{T}_{j\to i}}_{attraction} + \underbrace{\mathcal{B}_i\,\mathbf{e}_z}_{buoyancy} \quad . \tag{6}$$

We model the the fluid-particle interaction with a simple drag force with drag coefficients α_i. Neighboring particles $j \neq i$ act on particle i by forces $\mathbf{T}_{j\to i}$ that we suppose attractive and aligned with $\mathbf{r}_i - \mathbf{r}_j$. Due to gravity, there is a buoyancy term $\mathcal{B}_i\,\mathbf{e}_z$. The inertial term will be neglected in all what follows.

We suppose that buoyancy is dominant so that all particles will remain in the immediate vicinity of the surface. Different particles are similarly submerged in the fluid, so drag coefficients should be the same for all particles: we denote $\alpha_i = \alpha$. If the particles follow the motion of the interface, we can write

$$\mathbf{r}_i = \widehat{\mathbf{r}}_i + \boldsymbol{\eta}_i \quad , \quad \mathbf{v}_i = \widehat{\mathbf{v}}_i + \frac{d\boldsymbol{\eta}_i}{dt} \tag{7}$$

with $\boldsymbol{\eta}_i = \eta(\widehat{\mathbf{r}}_i,t)\mathbf{e}_z$ the surface elevation and $\widehat{\mathbf{r}}_i, \widehat{\mathbf{v}}_i$ the horizontal components of the particle's position and speed (we use hats for horizontal field components). By writing this, we ignore dynamic feedback of the particles on the wave. The horizontal motion is constrained by

$$\mathbf{0} \simeq \alpha\left(\widehat{\mathbf{u}}(\mathbf{r}_i,t)-\widehat{\mathbf{v}}_i\right) + \sum_{j\neq i}\widehat{\mathbf{T}}_{j\to i}. \tag{8}$$

In this balance, we suppose that the interaction forces $\sum_{j\neq i}\widehat{\mathbf{T}}_{j\to i}$ are dominant so that distances $||\widehat{\mathbf{r}}_i - \widehat{\mathbf{r}}_j||$ remain nearly fixed just as in a weakly deformable two-dimensional solid. We can decompose the particle speed as

$$\widehat{\mathbf{v}}_i \simeq \underbrace{\frac{d\widehat{\mathbf{r}}_g}{dt} + \omega\,\mathbf{e}_z \times (\widehat{\mathbf{r}}_i - \widehat{\mathbf{r}}_g)}_{dominant\ solid\ motion} + \underbrace{\frac{d\widehat{\mathbf{r}}'_i}{dt}}_{weak\ elastic\ motion} \tag{9}$$

separating the solid motion from a weak elastic motion. We introduce here $d\widehat{\mathbf{r}}_g/dt$, the horizontal speed of the center of mass $\mathbf{r}_g(t) = \sum_i \mathbf{r}_i/N$ and $\omega(t)$, the rotation speed of the raft. Elastic motions $d\widehat{\mathbf{r}}'_i/dt$ in the horizontal direction remain small whenever the raft is weakly compressible in the horizontal direction. To better know what this means, we estimate the order of magnitude of the elastic motion. With a fluid flow of order $\chi\Omega R$, the drag force can reach a magnitude $\alpha\chi\Omega R$. The drag force is balanced by an elastic force that brings particles back to equilibrium positions for which $\sum_{j\neq i}\widehat{\mathbf{T}}_{j\to i} = \mathbf{0}$. We can estimate the elastic force as $\kappa||\widehat{\mathbf{r}}'_i||$ with $\kappa = ||\widehat{\nabla}\widehat{\mathbf{T}}_{j\to i}||$ measuring the horizontal stiffness of the raft. The force balance leads to $||\widehat{\mathbf{r}}'_i|| = \alpha\chi\Omega R/\kappa$ as an order of magnitude for the elastic deviations and to $||d\,\widehat{\mathbf{r}}'_i/dt|| \sim \alpha\chi\Omega^2 R/\kappa$ for the elastic motion. Elastic motion can be ignored with respect to solid motion of order $||d\widehat{\mathbf{r}}_g/dt|| \sim \chi\Omega R$ whenever

$$\frac{\alpha\Omega}{\kappa} \ll 1. \tag{10}$$

The stiffer the raft in the horizontal direction (the higher κ), the smaller the elastic motion. We suppose that this condition is fulfilled and this allows us to ignore the weak elastic motion in all what follows.

Without elastic deviations, the motion of the raft is entirely determined by $\widehat{\mathbf{r}}_g(t)$ and $\omega(t)$ for which we can derive two simple equations. Summing (8) and $(\widehat{\mathbf{r}}_i - \widehat{\mathbf{r}}_g)\times$ (8) over all N particles, we can identify that

$$\frac{d\widehat{\mathbf{r}}_g}{dt} = \frac{1}{N}\sum_i \widehat{\mathbf{u}}(\widehat{\mathbf{r}}_i + \boldsymbol{\eta}_i, t) \tag{11a}$$

$$\omega = \frac{\sum_i [(\widehat{\mathbf{r}}_i - \widehat{\mathbf{r}}_g)\times \widehat{\mathbf{u}}(\widehat{\mathbf{r}}_i + \boldsymbol{\eta}_i, t)]\cdot \mathbf{e}_z}{\sum_i ||\widehat{\mathbf{r}}_i - \widehat{\mathbf{r}}_g||^2}. \tag{11b}$$

Due to Newton's third law ($\mathbf{T}_{j\to i} + \mathbf{T}_{i\to j} = \mathbf{0}$) and collinearity of $\mathbf{T}_{j\to i}$ and $\mathbf{r}_i - \mathbf{r}_j$ these relations are independent of the precise nature of the interactive forces. In both (11a) and (11b), we also note that it is necessary to evaluate the horizontal flow at the true particle position $\mathbf{r}_i = \widehat{\mathbf{r}}_i + \boldsymbol{\eta}_i$ on the surface. This subtlety is crucial to find the slow co-rotation of the raft.

We formulate a continuum limit for a circular raft of radius a, composed of many uniformly distributed particles. In the absence of flow, we suppose that the raft is centered on the origin of the cylinder. We then have

$$\frac{d\widehat{\mathbf{r}}_g}{dt} = \frac{1}{\pi a^2}\iint_{\mathcal{D}(t)} \widehat{\mathbf{u}}(\widehat{\mathbf{r}} + \boldsymbol{\eta}, t)\, d^2\widehat{\mathbf{r}} \tag{12a}$$

$$\omega = \frac{2}{\pi a^4}\iint_{\mathcal{D}(t)} [(\widehat{\mathbf{r}} - \widehat{\mathbf{r}}_g)\times \widehat{\mathbf{u}}(\widehat{\mathbf{r}} + \boldsymbol{\eta}, t)]\cdot \mathbf{e}_z\, d^2\widehat{\mathbf{r}} \tag{12b}$$

Here we denote $\mathcal{D}(t)$ is the domain where $||\widehat{\mathbf{r}} - \widehat{\mathbf{r}}_g|| \le a$. It is useful to rewrite the integrals of (12) using a translated coordinate system, centered on the raft. There we have

$$\frac{d\widehat{\mathbf{r}}_g}{dt} = \frac{1}{\pi a^2}\iint_{\mathcal{D}} \widehat{\mathbf{u}}(\widehat{\mathbf{r}} + \widehat{\mathbf{r}}_g + \widetilde{\boldsymbol{\eta}}, t)\, d^2\widehat{\mathbf{r}} \tag{13a}$$

$$\omega = \frac{2}{\pi a^4}\iint_{\mathcal{D}} [\widehat{\mathbf{r}}\times \widehat{\mathbf{u}}(\widehat{\mathbf{r}} + \widehat{\mathbf{r}}_g + \widetilde{\boldsymbol{\eta}}, t)]\cdot \mathbf{e}_z\; d^2\widehat{\mathbf{r}} \tag{13b}$$

Here we denote $\widetilde{\boldsymbol{\eta}}(\widehat{\mathbf{r}}, t) = \boldsymbol{\eta}(\widehat{\mathbf{r}} + \widehat{\mathbf{r}}_g, t)$ and $\mathcal{D}$ is now a stationary circular domain where $||\widehat{\mathbf{r}}|| \le a$. Since the flow is small and of order $O(\chi)$, we know that $\widehat{\mathbf{r}}_g, \boldsymbol{\eta} = O(\chi)$ too. This allows us to use Taylor expansions, to derive explicit formula for $\mathbf{r}_g$ and ω in different orders of χ.

C. First order motion: gyration . – To obtain a first order approximation for $\widehat{\mathbf{r}}_g$ and ω, we approximate

$$\begin{aligned} \widehat{\mathbf{u}}(\widehat{\mathbf{r}} + \widehat{\mathbf{r}}_g + \widetilde{\boldsymbol{\eta}}, t) &= \widehat{\mathbf{u}}(\widehat{\mathbf{r}}, t) + O(\chi^2) \\ &= \widehat{\nabla}\phi(\widehat{\mathbf{r}}, t) + O(\chi^2) \end{aligned} \tag{14}$$

in the integrals of (13a) and (13b). Using the vector identify $\widehat{\mathbf{r}}\times\widehat{\nabla}\phi = -\widehat{\nabla}\times(\widehat{\mathbf{r}}\phi)$ and integration theorems we can simplify the surfaces integrals to contour integrals.

$$\frac{d\widehat{\mathbf{r}}_g}{dt} = \frac{1}{\pi a^2}\int_0^{2\pi} a\,\phi(a,\theta,0,t)\,\mathbf{e}_r\, d\theta + O(\chi^2) \tag{15a}$$

$$\omega = \frac{2}{\pi a^4}\int_0^{2\pi} a^2\,\phi(a,\theta,0,t)\,\underbrace{(\mathbf{e}_r\cdot\mathbf{e}_\phi)}_{=0}\, d\theta + O(\chi^2) = O(\chi^2). \tag{15b}$$

This shows that the raft cannot rotate at first order, we can only have a translational motion. After some calculations, we find

$$\widehat{\mathbf{r}}_g = \rho\;(\cos\Omega t\,\mathbf{e}_x + \sin\Omega t\,\mathbf{e}_y) + O(\chi^2), \tag{16}$$

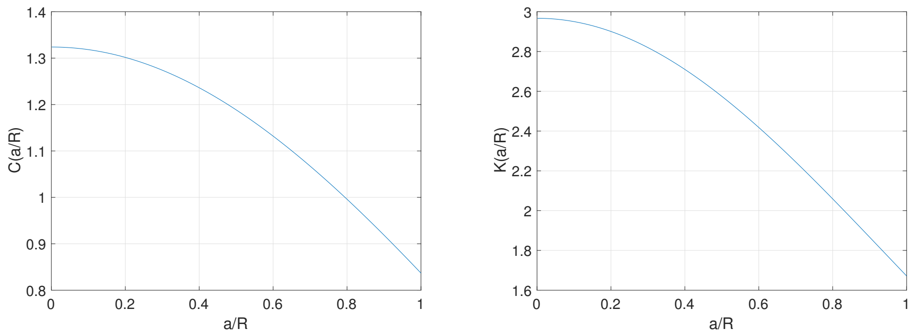

Fig. 1: The gyration radius of the raft is $\rho = \chi R\, C(a/R)$ and the rotation speed for the co-rotation is $\overline{\omega} = \chi^2 \Omega\, K(a/R)$. Here we show $C(a/R)$ and $K(a/R)$ as functions of the non-dimensional radius a/R of the raft. We fix $H/R = 2.17$ as in the experiment.

with

$$\rho = \chi R \underbrace{\frac{2R}{a}\frac{1}{(k_1^2-1)}\frac{J_1(k_1 a/R)}{J_1(k_1)}}_{C(a/R)} . \tag{17}$$

We call this motion the gyration of the raft: the center of mass of the raft rotates with time, along with the wave. We denote ρ the gyration radius that scales as $\rho \sim \chi R$. The coefficient of proportionality $C(a/R)$ is shown in figure 1 as a function of a/R. It varies from 1.3 to 0.8 for a/R varying from 0 to 1.

D. Second order motion: co-rotation . – To describe the motion of the raft up to second order, we approximate

$$\widehat{\mathbf{u}}(\widehat{\mathbf{r}}+\widehat{\mathbf{r}}_g+\widetilde{\boldsymbol{\eta}},t) \quad = \quad \widehat{\nabla}\phi\,(\widehat{\mathbf{r}},t)+[(\widehat{\mathbf{r}}_g+\boldsymbol{\eta})\cdot\nabla]\widehat{\nabla}\phi(\widehat{\mathbf{r}},t)+\widehat{\widehat{\mathbf{u}}}(\widehat{\mathbf{r}})+\widehat{\mathbf{u}}'(\widehat{\mathbf{r}},t)+O(\chi^3) \tag{18}$$

in (13a). In the right hand side, we see a first order Taylor expansion of the potential flow. $\widehat{\boldsymbol{r}}_g$ can be replaced with (16) and we can also simplify $\widetilde{\boldsymbol{\eta}}(\widehat{\mathbf{r}},t) = \boldsymbol{\eta}(\widehat{\mathbf{r}},t)+O(\chi)$. The second order flow correction $\widehat{\widehat{\mathbf{u}}}+\widehat{\mathbf{u}}'$ is also included. We focus on the stationary terms that can induce a slow mean motion. Using bars to denote time-independent fields, we can find that

$$\begin{aligned}\overline{\widehat{\mathbf{u}}(\widehat{\mathbf{r}}+\widehat{\mathbf{r}}_g+\widetilde{\boldsymbol{\eta}},t)} &= \widehat{\widehat{\mathbf{u}}}(\widehat{\mathbf{r}})+\overline{(\boldsymbol{\eta}\cdot\nabla)\widehat{\nabla}\phi\,(\widehat{\mathbf{r}})}+O(\chi^3) \qquad (19)\\ &= \widehat{\widehat{\mathbf{u}}}(\widehat{\mathbf{r}})+\frac{\Omega R^2}{2\,r}\left(\frac{2\chi k_1}{(k_1^2-1)^2}\frac{J_1(k_1 r/R)}{J_1(k_1)}\tanh(k_1 H/R)\right)^2\mathbf{e}_\theta+O(\chi^3).\end{aligned}$$

Next to the steady streaming flow for which we have no analytical expression, we find a Stokes drift correction that can be explicitly calculated. We admit that $\widehat{\mathbf{r}}_g(t,\tau)$ can have a dependance on a slow time-scale $\tau = (\chi\Omega)^{-1}$. We then find that

$$\frac{d\widehat{\mathbf{r}}_g}{d\tau} \quad \simeq \quad \frac{1}{\pi a^2}\iint_{\mathcal{D}}\left[\widehat{\widehat{\mathbf{u}}}(\widehat{\mathbf{r}})+\overline{(\boldsymbol{\eta}\cdot\nabla)\widehat{\nabla}\phi\,(\widehat{\mathbf{r}})}\right]d^2\widehat{\mathbf{r}} = \mathbf{0} \tag{20}$$

due to axisymetry. The gyration center of an initially centered raft will remain close to the origin on timescales $(\chi\Omega)^{-1}$. For the stationary component $\overline{\omega}$ of the rotation speed (13b) we find up to second order

$$\overline{\omega} = \frac{2}{\pi a^4}\int_0^a\int_0^{2\pi}\left[\widehat{\widehat{\mathbf{u}}}(\widehat{\mathbf{r}})+\overline{(\boldsymbol{\eta}\cdot\nabla)\widehat{\nabla}\phi\,(\widehat{\mathbf{r}})}\right]_\theta r^2\,dr\,d\theta. \tag{21}$$

The study of Bouvard et al. [4] suggested that the steady streaming flow has a weak azimuthal component. If we ignore these contributions ($\overline{u}_\theta \approx 0$), we can calculate

$$\overline{\omega} \simeq \Omega\chi^2 \underbrace{\frac{R^2}{a^2}\frac{4k_1^2}{(k_1^2-1)^2}\tanh^2(k_1 H/R)\frac{J_1^2(k_1 a/R) - J_0(k_1 a/R)J_2(k_1 a/R)}{J_1^2(k_1)}}_{K(a/R,H/R)}. \tag{22}$$

As shown in figure 1, K varies from 2.97 to 1.67 for a/R varying from 0 to 1 in our set-up with $H/R = 2.17$. A bigger raft rotates slower. The value 2.97 for very small rafts coincides with Stokes drift rotation speed at the center. In the experiments we found $K \simeq 2$, which is compatible with this result.

E. Boundary layer effects: counter-rotation. – To describe the counterrotating motion of large rafts, we must take into account that such large rafts reach into the boundary layer region while they gyrate. We perform all calculations in the frame attached to the cylinder. We approximate

$$\widehat{\mathbf{u}}(\widehat{\mathbf{r}}+\boldsymbol{\eta},t) = \widehat{\nabla}\phi - \widehat{\nabla}\phi|_{r=R}\, e^{-(R-r)/\delta} + O(\chi^2). \tag{23}$$

in (12b). The effect of potential flow is already known up to second order and induces the co-rotation $\overline{\omega}$. For large rafts, we need to correct the slow rotation speed as

$$\overline{\omega}_{tot} = \overline{\omega} + \overline{\omega}_{BL}, \tag{24}$$

where $\overline{\omega}_{BL}$ contains the stationary counter-rotation caused by the boundary layer correction alone. We can calculate

$$\omega_{BL} = \frac{2}{\pi a^4}\iint_{\mathcal{D}(t)} [(\widehat{\mathbf{r}} - \widehat{\mathbf{r}}_g)\times(-\widehat{\nabla}\phi|_{r=R}\, e^{-(R-r)/\delta})]\cdot \mathbf{e}_z\, d^2\widehat{\mathbf{r}} \tag{25}$$

and the time-average of this yields $\overline{\omega}_{BL}$. To evaluate this integral, we parametrize the time-dependent region $\mathcal{D}(t)$ that is occupied by the raft . If the gyration radius is small compared to the size of the raft ($\rho \ll a$), we can approximate

$$\mathcal{D}(t)\ :\ r \in [0, a + \rho\cos(\theta - \Omega t)] \quad , \quad \theta \in [0, 2\pi[\tag{26}$$

up to errors of $O(\rho^2/a^2)$. Using the definition (16) of the gyration radius ρ, we then express $\widehat{\mathbf{r}} - \widehat{\mathbf{r}}_g$ in cylindrical components to find

$$\omega_{BL} \simeq -\frac{4\Omega R\chi}{\pi a^4 (k_1^2-1)}\int_0^{2\pi}\int_0^{a+\rho\cos(\theta-\Omega t)} (r - \rho\cos(\theta - \Omega t))\cos(\theta-\Omega t)\, e^{(r-R)/\delta}\, r\, dr\, d\theta. \tag{27}$$

Due to the presence of the exponential factor, the integrandum rapidly decays away from the boundary $r = R$, which allows some simplifications. We introduce a change of variables $s = (r - R)/\delta$ and approximate the bound $r = 0$ by $s = -R/\delta \to -\infty$. In the integrandum, we also approximate all other occurrences of $r \simeq R$. Integration over s is then very simple, giving

$$\omega_{BL} = -\frac{4\Omega R^2\delta\chi}{\pi a^4(k_1^2-1)}\, e^{(a-R)/\delta}\int_{0-\Omega t}^{2\pi-\Omega t}\left(R - \rho\cos\widetilde{\theta}\right)\cos\widetilde{\theta}\, e^{(\rho/\delta)\cos\widetilde{\theta}}\, d\widetilde{\theta}$$

with $\widetilde{\theta} = \theta - \Omega t$. This integral can be evaluated analytically in terms of modified Bessel functions I_m as we have elementary integrals

$$\int_{0-\beta}^{2\pi-\beta} e^{\zeta\cos\widetilde{\theta}}\, d\widetilde{\theta} = 2\pi I_0(\zeta) \tag{28a}$$

$$\int_{0-\beta}^{2\pi-\beta} \cos\widetilde{\theta}\, e^{\zeta\cos\widetilde{\theta}}\, d\widetilde{\theta} = 2\pi I_1(\zeta) \tag{28b}$$

$$\int_{0-\beta}^{2\pi-\beta} \cos^2\widetilde{\theta}\, e^{\zeta\cos\widetilde{\theta}}\, d\widetilde{\theta} = \pi(I_0(\zeta) + I_2(\zeta)) \tag{28c}$$

$\forall\beta \in \mathbb{R}$. The first relation is well known. The second and third relation can be obtained by deriving the first relation with respect to ζ and using recurrence relations of modified Bessel functions. With this, we obtain

$$\omega_{BL} = -\chi\Omega\frac{8\delta R^3}{a^4}\frac{e^{(a-R)/\delta}}{(k_1^2-1)}\left\{I_1\left(\frac{\rho}{\delta}\right) - \frac{\rho}{2R}\left[I_0\left(\frac{\rho}{\delta}\right) + I_2\left(\frac{\rho}{\delta}\right)\right]\right\}.$$

We notice that ω_{BL} is time-independent and since we have $\rho \ll R$, the first term proportional to $I_1(\rho/\delta) > 0$ dominates. Therefore, we can expect a counter-rotation. It is useful to remember that this formula only makes sense when $\rho, \delta \ll R$, $\rho + a \leq R$ and when the wave-magnitude remains small. In the limit $\rho \ll \delta$, we can use a Taylor expansion $I_1(z) \simeq z/2$ and with $\rho \ll R$ we can ignore the contributions from I_0 and I_2. This then yields

$$\omega_{BL} \simeq -\chi^2\Omega\frac{4R^4}{a^4}\frac{e^{(a-R)/\delta}}{(k_1^2-1)}C(a/R),$$

with $C(a/R)$ defined in (17). We notice that the counter-rotation is of order $\sim \chi^2$ just as the co-rotation.