NARROW SHIP WAKES AND WAVE DRAG FOR PLANING HULLS

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The angle formed by ship wakes is usually found equal to its Kelvin value, $\alpha = 19.47$ degrees. However we recently show that this angle can be significantly smaller at large Froude number [8]. We show how the limited range of wave numbers excited by the ship explains the observed decrease of the wake angle as $1/\mathrm{Fr}$ for $\mathrm{Fr} > 0.5$, where $\mathrm{Fr} = U/\sqrt{gL}$ is the Froude number based on the hull length L. At such large Froude numbers, sailing boats are in the planing regime, for which the wave drag becomes a decreasing function of the velocity. We discuss here the possible connection between the evolutions of the wake angle and wave drag at large Froude number.

NOMENCLATURE

Symbol	Definition	(unit)
B	Waterline beam	(m)
$\mathbf{c}_\varphi$	Phase velocity	(m s^{-1})
$\mathbf{c_g}$	Group velocity	(m s^{-1})
C_W	Wave-making coefficient	
D	Static immersed volume	(m^3)
Fr	Hull Froude number	
g	Acceleration of gravity	(m s^{-2})
$\mathbf{k}$	Wave number	(m^{-1})
L	Waterline length	(m)
P	Pressure	(N m^{-2})
R_W	Wave-making resistance	(N)
$\mathbf{U}$	Boat velocity	(m s^{-1})
α	Half-angle of the wake	
θ	Angle $(\mathbf{k}, \mathbf{U})$	
ρ	Density of water	(kg m^{-3})

1 INTRODUCTION

A ship moving on calm water generates gravity waves with a characteristic V-shaped pattern. Lord Kelvin in 1887 [4] was the first to explain this phenomenon and to show that the wedge angle is constant, independent of the boat velocity. According to this classical analysis, only the wavelength and the amplitude of the waves change with the velocity and the half-angle of the wedge remains equal to 19.47 degrees.

In contrast to this result described in many textbooks, we have shown recently that the wake angle is no more constant at large velocity [8] and decreases as $1/U$. We have shown that this decrease can be modeled by including the finite length of the boat in Kelvin's analysis.

Some years before Kelvin's work, William Froude, by towing model boats, observed that the hydrodynamic drag increases rapidly with the boat speed U, and more precisely that the drag is a function of the hull Froude number $\mathrm{Fr} = U/\sqrt{gL}$, where L is the waterline length. Following the pioneering works of Froude [4], Michell [7, 10] and Havelock [5], the computation of hydrodynamic drag still represents a challenge for naval architects. The wave drag or wave-making resistance R_W is the part of this hydrodynamic drag that corresponds to the energy radiated by the waves generated by the hull translation. For a displacement hull sailing at large velocity (Froude number in the range 0.2 to 0.5) the major part of the hydrodynamic drag is given by the wave drag.

In this paper we discuss the possible link between the decrease of the wake angle observed at large Froude number and the evolution of the wave drag for planing sailing boats.

2 WAVE PATTERN

When a boat sails on calm water at constant velocity U, the waves present around and behind the hull are only those that are stationary in the frame of reference of the boat. For a given wave of wave number k propagating in the direction θ with respect to the boat course, this condition writes:

$$U \cos\theta(k) = c_\varphi(k) \tag{1}$$

where $c_\varphi(k)$ is the phase velocity of the considered wave (figure 1).

Because of the dispersive nature of gravity waves, c_φ is function of the wave number, $c_\varphi = \sqrt{g/k}$, implying that for a given propagation direction θ only one wavenumber is selected by Eq. 1:

$$k(\theta) = \frac{g}{U^2 \cos^2\theta}. \tag{2}$$

As a consequence, the smallest wave number (i.e. the largest wave length) compatible with the stationary condition is given by $k_g = g/U^2$, and corresponds to waves propagating in the boat direction ($\theta = 0$). These so-called transverse waves are visible along the hull and following the boat.

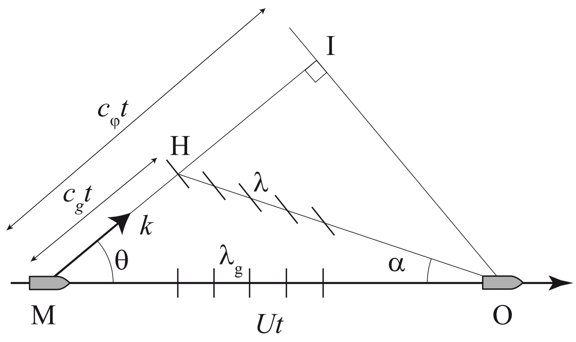

Figure 1: Geometric construction of the wave pattern and angle definitions for a boat sailing at constant velocity U.

Importantly, energy propagates at the group velocity and not at the phase velocity, and for gravity waves the group velocity is equal to half the phase velocity ($\mathbf{c_g} = \frac{1}{2}\mathbf{c}_\varphi$) [6]. It follows from this 1/2 factor that the angle α, where waves of a given wave number are observed (figure 1), is given by [8]:

$$\alpha(k) = \tan^{-1}\left(\frac{\sqrt{k/k_g - 1}}{2k/k_g - 1}\right). \tag{3}$$

This evolution of the angle α with the wave number is shown in figure 2.

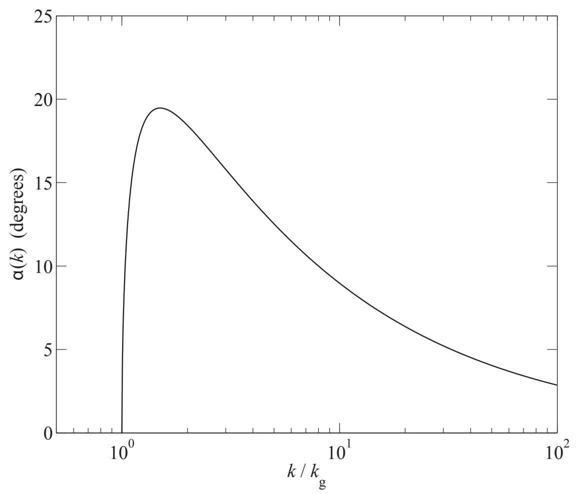

Figure 2: Evolution of the angle α versus the wave number ratio k/k_g (Eq. 3), where $k_g = g/U^2$ is the gravity wave number.

This plot shows that for any given angle α smaller than 19.47 degrees there are two possible values of k that correspond to two directions θ (Eq. 2). One solution corresponds to transverses waves (smaller θ) and the other to divergent waves (larger θ). The angle α takes its maximum value $\alpha_0 = 19.47$ deg for $k_0/k_g = 3/2$, and no waves can be observed beyond this angle. This maximum wake angle corresponds to a cusp (a caustic) in the wave pattern, and also to the locus of maximum amplitude of the waves, since $\partial\alpha/\partial k = 0$, which implies an accumulation of energy at $k_0/k_g = 3/2$. These results correspond to the well known Kelvin angle [3].

In this classical description the boat is considered as a point source, generating all the waves with a small constant amplitude (broad band flat spectrum). In reality all the points of the hull are sources and the detail of the amplitude of the wave depend of the exact shape, trim, sinkage of the hull and of the Froude number. For example, for a poorly streamlined hull at low Froude number, two V-shaped wakes are visible, one originating at the bow and the other at the stern. The waves generated by the boat are therefore characterized by a spectrum which cannot be considered as flat, and the resulting wake pattern may escape from the classical Kelvin's description.

3 WAVE ANGLE FOR RAPID BOATS

We recently showed that the commonly admitted result of Kelvin of a constant wake angle equal to 19.47 degrees is no longer true at large velocity for planing boats [8]. This is illustrated in figure 3, showing a wake angle significantly smaller than the Kelvin prediction.

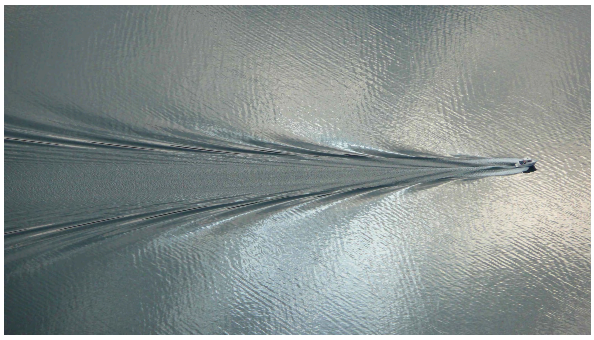

Figure 3: Photograph of a fast planing motorboat exhibiting a narrow wave wake (source: http://en.wikipedia.org/wiki/Wake).

Analyzing a set of airborne images from Google Earth©, we measured the wake angles and the Froude numbers for boats of various sizes and velocities. Using the scale provided on the images, we measured the overall length of the boat (assumed to be equal to the waterline length L) and the wavelength of the waves on the edge of the wake. From this wavelength the boat velocity U is determined using Eq. 2 and the Froude number is then computed. Our data clearly show a decrease of the wedge angle for Froude numbers larger than 0.5 (figure 2 of [8]). Values as small as 7 degrees are observed.

Wake angles smaller than the Kelvin prediction can be explained as follows. The key argument is that, contrary to the Kelvin assumption, a moving boat does not excite all the wavelengths with the same energy. In particular it cannot excite surface waves significantly larger than its waterline length L. The energy radiated by the boat is therefore characterized by a spectrum which is truncated below the wavenumber $k_{min} \sim 2\pi/L$. At large boat velocity this wavenumber can be larger than the wave number k_0 which corresponds to the maximum Kelvin angle in figure 2. Thus only the wave numbers corresponding to divergent waves, i.e. rightmost part of figure 2, are of significant amplitude, so the largest visible angle is given by Eq. 3 taken for $k = k_{min} \sim 2\pi/L$. This model predicts that the wake angle is given by the Kelvin prediction as long as the k_0 mode contains energy, i.e. up to $\mathrm{Fr}_c = \sqrt{3/4\pi} = 0.49$, and by a decreasing function $\alpha(k = 2\pi/L)$ at larger Froude. For $\mathrm{Fr} \gg \mathrm{Fr}_c$ the wake angle decreases as

$$\alpha \approx \frac{1}{2\sqrt{2\pi}\mathrm{Fr}}. \quad (4)$$

This previously unnoticed Froude number dependence of the wake angle compares well with the wake angles observed from airplane images. This is also consistent with the fact that at $\mathrm{Fr} > 0.5$ the transverse waves behind the boat ($\theta = 0$), which are visible for smaller Froude numbers, are no more visible (see figure 3), since they fall outside the wave spectrum excited by the boat. Equation 4 is also found to describe very well the wave patterns obtained by numerical simulations (figure 4). More details on the numerical simulation can be found in Ref. [8].

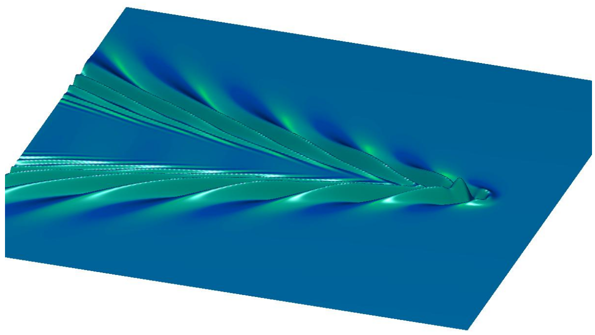

Figure 4: Perspective view of the wave pattern generated by an axisymmetric (Gaussian) pressure distribution at $\mathrm{Fr} = 1$. The measured wake angle is $\alpha = 11$ degrees.

4 WAVE DRAG

In order to describe the classical result of the increase of the wave drag for displacement navigation ($\mathrm{Fr} < 0.5$) we come back to figure 1. We focus here on the transverse waves propagating in the boat direction ($\theta = 0$). These waves are the stationary waves observed along the side of the hull and behind the boat. Their wavenumber is given by Eq. 2, $k_g = g/U^2$, and their wavelength $\lambda_g = 2\pi/k_g$ can be written as $\lambda_g = 2\pi L\, \mathrm{Fr}^2$. For increasing speed their wavelength increases, up to a particular velocity for which the wavelength is equal to the length of the boat. This velocity corresponds to $\mathrm{Fr} = 1/\sqrt{2\pi} \approx 0.4$. For this value the waves generated by the bow are in phase with the ones emitted at the stern and the draught or sinkage of the hull is maximum. This critical velocity is known as the hull limit speed, because around this Froude number the wave drag increases drastically and the trim of the boat starts to be strongly affected by the waves it generates. We now know that this "limit speed" can be overcome with light and powerful boats as they reach the planing regime. In this regime of large Froude number, hydrodynamic lift becomes significant, decreasing the immersed volume of the hull. Because of the resulting smaller mass of fluid which needs to be pushed away, a decrease of the wave drag is observed. During this transition to planing, a significant acceleration of the boat can be observed. We discuss here the possible connection between this wave drag decrease during planing and the decrease of the visible wake angle described in the previous section.

The wave drag R_W is the part of the hydrodynamic drag due to the energy radiated by the waves generated by the boat. In order to compare boats of different forms and displacement a dimensionless wave drag coefficient C_W is usually defined. Assuming hulls having all the same shape but not the same size, the wave drag will only depend of the boat velocity U, waterline length L, gravity g and water density ρ. One finds by dimensional analysis:

$$\frac{R_W}{\rho U^2 L^2} = C_W(\mathrm{Fr}). \quad (5)$$

In reality this coefficient C_W also depends on the exact shape of the boat, and alternate definitions where L^2 is replaced by LB or B^2 (where B is the beam of the hull) are also found in the literature. Another possibility is to build a dimensionless drag coefficient by normalizing the wave drag force R_W by the weight of the boat $\rho g D$, where D is the static immersed volume of the hull. For displacement boat, the wave drag coefficient rapidly increases (at least as Fr^4 if defined by Eq. 5) and becomes the dominant part of the hydrodynamical drag at large Fr. Note that the power law $C_W \propto Fr^4$ can be recovered by scaling argument, assuming that the amplitude of the waves scales as U^2 (using Bernoulli relation) and that the wavelength observed along the Kelvin angle scales as U^2 (Eq. 2).

In order to compute the wave drag, Havelock [5] has introduced a classical simplification which consists in replacing the boat by an imposed pressure field $P(x, y)$ at the water surface. The resulting surface deformation $\zeta(x, y)$ can then be computed as a Fourier integral (see Eq. 2.17b of Ref. [9], or Eq. 11 of Ref. [1]). From this imposed pressure and calculated wave field, the wave drag is then computed by integrating the product of the local pressure by the slope of the interface in the direction of the motion:

$$R_W = \iint P(x, y) \frac{\partial \zeta}{\partial x} dxdy. \quad (6)$$

On figure 4 we have simulated the wave pattern generated by a moving Gaussian pressure field, $g(r) = (2\pi F_0/L^2) \exp\left(-2\pi^2 r^2/L^2\right)$, where F_0 is a normalization force, which corresponds here to the weight of the boat ($F_0 = \rho g D$). From this simulated surface height, we have computed the wave drag using Eq. 6 for various Froude numbers. The results, plotted in figure 5, are in perfect agreement with the exact result found by Benzaquen *et al.* [1] for a Gaussian pressure field:

$$C_W = \left(\frac{D}{L^3}\right)^2 \frac{1}{\mathrm{Fr}^8} \int_0^{\pi/2} \frac{d\theta}{\cos^5\theta \exp\left[\left(\sqrt{2\pi}\mathrm{Fr}\cos\theta\right)^{-4}\right]} \tag{7}$$

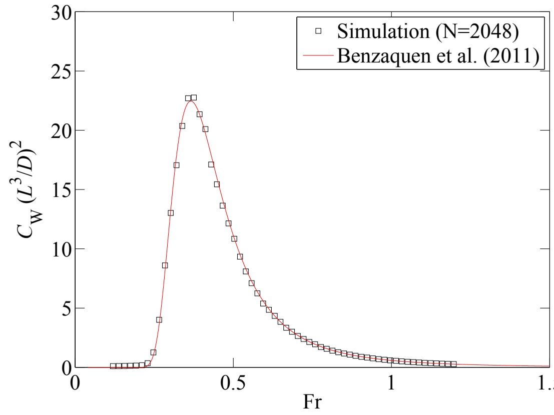

Figure 5: Dimensionless wave drag calculated for a gaussian moving pressure field with our simulated wave field (□) and comparison with Eq. 7 (—).

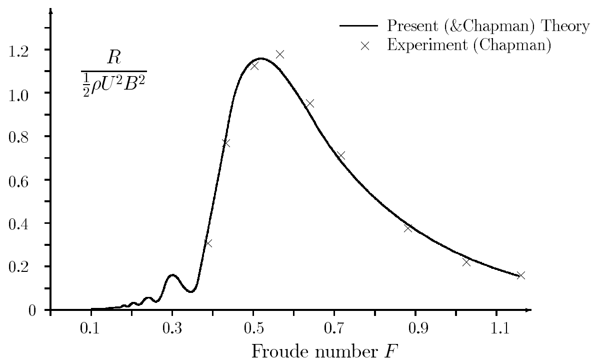

Figure 6: Dimensionless wave drag for a parabolic strut (figure 1 of Ref. [11].

This wave drag coefficient is maximum for Fr $\simeq 0.37$, followed by a decrease as $C_W \simeq 1/\mathrm{Fr}^4$ at large Froude numbers. Interestingly, this maximum is very close to the critical Froude number $\mathrm{Fr}_c \simeq 0.49$ at which the wake angle starts decreasing. Both results are consequence of the finite extent of the wave spectrum excited by the disturbance: as the Froude number is increased, the surface deformation in the vicinity of the boat is no longer able to supply energy to the waves of wavelength $\lambda_g = 2\pi U^2/g$, resulting in a combined decrease of the wake angle ($\alpha \simeq 1/\mathrm{Fr}$) and of the wave drag ($C_W \simeq 1/\mathrm{Fr}^4$).

The overall shape of C_W computed by Eq. 7 is surprisingly similar to the experimental curve of Chapman [2] with computation by Tuck *et al.* [11] (figure 6). This curve is usually interpreted as the result of the lift of the hull and the resulting decrease of the immersed volume at Fr > 0.5. However, in our analysis, the prescribed pressure $P(x, y)$ does not depend on the velocity, so it does not contain the physics of the dynamical lift on the hull. This suggests that the dynamics of the planing and the decrease of the immersed volume are not necessary ingredients for the decrease of the wave drag at large Froude number. Note that the decrease of the wave drag at large velocity is often partly hidden by the increase of the other sources of hydrodynamic drag, which increase as Fr^2.

5 CONCLUSIONS

At large velocity many racing sailing boats are now planing under the action of the strong hydrodynamic lift. The fact that the dynamically immersed volume is smaller than in static condition provides a reasonable argument for the diminution of the wave drag. We propose here an alternative interpretation, in which the combined decrease of the wave drag and the wake angle both follow from the finite extent of the wave spectrum excited by the ship. This interpretation is based on our simulations of the wave pattern generated by an imposed pressure disturbance, suggesting that the narrow wake angles at large Froude number can be observed without lift and thus without planing regime. Further investigations are necessary to better describe the relative importance of trim and sinkage evolution of planing boat to better understand the relative importance of the finite size of the boat compared to dynamic lift.

We note that the present description is by construction limited to stationary motion, i.e. boat translating on a flat sea surface. In real situations, when in planing conditions the wind and thus the wind waves are usually large, inducing a periodic motion of the boat at the wave encounter frequency. This non stationarity increases the hydrodynamic drag when sailing at close reach but can also decreases the drag when surfing on swell.

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