LETTERS

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Hydrodynamic dispersion broadening of a sedimentation front

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Hydrodynamic dispersion is responsible for the spreading of the sedimentation front even in a noncolloidal monodisperse suspension. Measurements of the broadening of the top front observed during sedimentation have been used in determining the hydrodynamic dispersion coefficient. Hindered settling has an opposed effect and leads to the self-sharpening of the front. Both effects have to be taken into account simultaneously. This Letter provides a simple, but complete determination of the space and time concentration profile and shows that the final front should consist of a steady-shape profile propagating at constant velocity. With such a solution, the data of Davis *et al.* [AIChE J. **34**, 123 (1988); J. Fluid Mech. **196**, 107 (1988)] give hydrodynamic dispersion coefficient five times larger than their former analysis, in agreement with Lee *et al.* [Phys. Fluids A **4**, 2601 (1992)].

In a noncolloidal suspension of monodisperse sedimenting particles, the motion of an individual particle is influenced by the relative positions and velocities of surrounding particles. Thus the velocity of an individual particle varies around the mean settling velocity $U(C)$ of the suspension of concentration C. This so-called hydrodynamic dispersion phenomenon[1–5] has stimulated both experimental[6,7] and theoretical[4,5,8] works. Experimental determination of the related hydrodynamic dispersion coefficient $D(C)$ has been performed in two directions: The first deals with tracking the velocity fluctuations of either a tagged particle settling among and being part of the homogeneous suspension[3] or the whole suspension;[6] the variance of these fluctuations gives D. The other one takes advantage of the large concentration gradient occurring at the top of a sedimenting suspension, i.e., the sedimenting shock front spreads due to the dispersion process.[2,7] We will address here the applicability of the latter method. The basic equation governing the sedimentation process is a convection–diffusion equation,

$$\frac{\partial C}{\partial t}+\frac{\partial\,[CU(C)]}{\partial x}=\frac{\partial[D(C)\partial C/\partial x]}{\partial x}, \tag{1}$$

where x is the vertical coordinate, $C(x,t)$ the space and time dependent concentration, $Q=CU(C)$ the particle flux. The left-hand side (LHS) of (1) is the hyperbolic convection equation accounting for the nondiffusive sedimentation process. As $U(C)$ is a decreasing function of C, this LHS leads to a shock front[9,10] (kinematic wave) propagating with a velocity $V_s=U(C_0)$; the sharp front separates the homogeneous suspension of initial concentration C_0 from the clear fluid ($C=0$) at the top. Hydrodynamic dispersion modifies the above picture and spreads out the front. One way to analyze this effect is to superimpose the diffusion on the sharp front. In other words, the spreading is measured from the broadening of the shock front. This is analogous to replacing the LHS of (1) by $\partial C/\partial t+V_s\,\partial C/\partial x$, the solution of which is well known,

$$C(x,t)=C_0\{1-\mathrm{erf}[(x-V_st)/\sqrt{(4Dt)}]\}/2, \tag{2}$$

where a constant dispersion coefficient is assumed and concentrations at the boundaries are equal to 0 and C_0 for $x=+\infty$ and $x=-\infty$, respectively. Davis *et al.*[2] used this approach to measure D. Lee *et al.*[7] pointed out that self-sharpening and dispersion have to be taken into account simultaneously [Eq. (1) is not linear]. They determined the derivatives involved in Eq. (1) directly from their experiments and obtained dispersion coefficients five times larger than Davis *et al.*[2] Lee *et al.* explained the discrepancy by the improper use of Eq. (2). We propose a complete analysis of the front behavior, which reconciles the two authors and leads to the front profile, i.e., the space and time dependence of the concentration along the sedimentation column.

Basically, self-sharpening and dispersion broadening have opposing effects on the front, thus they can balance each other, leading to a stabilized dispersed profile: After a transient, the front profile will propagate with a constant velocity V_s, without changing its shape. Indeed this remark is straightforward if we note that Eq. (1) is, for constant D and quadratic particle flux $Q(C)$, a Burger's type equation,[10] the solution of which is a stationary profile in a reference frame

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traveling at the constant velocity V_s. The physical interpretation of this wave point of view is that in the moving frame of reference, there is a dynamic equilibrium: the net convective flux exactly balances the dispersion flux. This was basically Einstein's argument to derive the diffusion coefficient of a colloidal suspension.[11]

For any flux, provided it leads to self-sharpening,[9] we can look for traveling solutions of Eq. (1) of the form $C(X)$, $X=x-V_st$,

$$C[U(C)-V_s]-D(C)\frac{\partial C}{\partial X}+A=0. \tag{3}$$

The constant terms A and V_s are determined by the boundary conditions at $+\infty$ and $-\infty$, where the concentration tends to C_2 and C_1 and the concentration gradients vanish, giving

$$V_s=[C_2U(C_2)-C_1U(C_1)]/(C_2-C_1), \tag{4}$$

$$A=C_2C_1[U(C_2)-U(C_1)]/(C_2-C_1). \tag{5}$$

For the top front between clear fluid ($C_2=0$) and an initially homogeneous suspension of concentration $C_1=C_0$, we get $A=0$ and $V_s=U(C_0)$, the classical shock front of Kynch approach.[9] Moreover Eq. (3) is integrable, leading to the stationary front shape

$$x-V_st=\int D(C)/\{C[U(C)-V_s]+A\}dC. \tag{6}$$

This equation can be easily computed, provided $D(C)$ and $U(C)$ are known. But this traveling steady state occurs only after a transient time, during which the front shape is time dependent (initial shape: at $t=0$, straight jump at $x=0$, from $C_2=0$ for $x>0$ to $C=C_1$ for $x<0$).

Indeed, in real experiments, in order to determine the dispersion coefficient, only the early stage of the sedimentation process requires being analyzed because of the residual polydispersity of the suspension.[3,7] Thus the transient issue has to be considered. The full problem is computable for any type of flux Q. Moreover there is an analytical solution of the problem if a quadratic flux is assumed; in this case, Eq. (1) becomes a Burger's equation[10] which captures the essential features of the problem. Writing this flux, $Q=\alpha C^2+\beta C$, the propagating velocity of the concentration C, $W(C)=2\alpha C+\beta$, is linear in C. The velocities corresponding to the boundary concentrations are $w_1=W(C_1)$ and $w_2=W(C_2)$. The constant drift velocity of the profile $V_s=(w_1+w_2)/2$, is in agreement with the general case [Eq. (4)]. The profile, $C(x,t)$, is given for all times by

$$C(x,t)=C_1+(C_2-C_1)/\{1+h(x,t)\times\exp[(w_2-w_1)(x-V_st)/2D]\}, \tag{7}$$

where

$$h(x,t)=\frac{\{1+\mathrm{erf}[(x-w_2t)/\sqrt{4Dt}]\}}{\{1-\mathrm{erf}[(x-w_1t)/\sqrt{4Dt}]\}}. \tag{8}$$

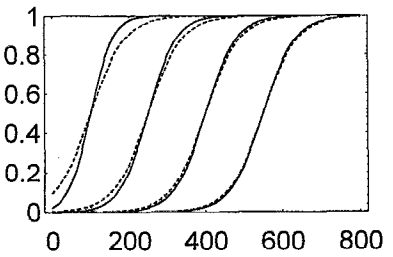

FIG. 1. Reduced concentration $C(x,t)/C_0$ versus the reduced position X at different reduced times $T=100$, 200, 300, 400. Solid lines are the transient solution [Eq. (7)], dashed lines are the stationary solution [Eq. (9)].

For large enough time and a given ratio x/t in the range $w_1<x/t<w_2$, $h(x,t)$ tends to unity and the solution approaches the steady-shape profile propagating at the constant drift velocity V_s

$$C(X)=C_1+(C_2-C_1)/\{1+\exp[(w_2-w_1)X/2D]\}. \tag{9}$$

This solution is a particular case of the general one [Eq. (6)]. We can give a quantitative criterion to decide under which conditions the stationary solution applies. At large t, the arguments of the error functions are small and we can develop the function $h\sim 1+2X\sqrt{\pi Dt}$. The stationary front occurs for $(w_2-w_1)X/2D$ between -1 and $+1$; then stationary profile requires large t compared to $t_c=16D/[\pi(w_2-w_1)]^2$. The profiles described by (7) and (9) are almost S shaped (Fig. 1). This shape is often mistaken for an error function shape [Eq. (2)].

The application of the above to our problem is straightforward if we restrict our attention to the low concentration limit and a constant diffusion coefficient D. Experimental data[2,3,7] satisfied the latter condition, except in the very close vicinity of $C=0$ ($C<2\%$), so only the near-zero part of the profile is questionable. The former assumption permits a first order approximation in C, of the expression $U(C)=V_0(1-C)^p\sim V_0(1-pC)$, where V_0 is the Stokes settling velocity of a single particle. Hence, the particle flux is quadratic. Initially, the suspension is homogeneous at concentration C_0, then $C_1=C_0$ and $C_2=0$, $w_1=V_0(1-2pC_0)$ and $w_2=V_0$; the drift velocity is $V_s=U(C_0)$.

Let us plot the profiles in the range of values corresponding to the measurements of Lee *et al.*[7] (Figs. 2 and 3 of Ref. 7). We take an initial concentration $C_0=10\%$, then

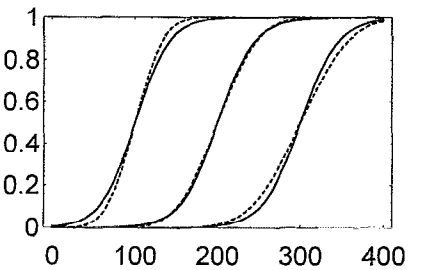

FIG. 2. Reduced concentration profile versus X at different times $T=100$, 200, 300. Solid lines transient solution [Eq. (7)]; dashed lines, best fit to the exact solution with the error function solution with the apparent dispersion coefficient D_{ap}.

$V_s = U(C_0) = V_0/2$, for $p = 5$ and a constant dispersion coefficient $D = 22aU(C_0)$ (a, spherical particle radius). Using the reduced variables, $X = x/a$, $T = V_s t/a$, the typical values corresponding to the experiment ($t = 332s$, $a = 68$ μm, $U_0 = 10^{-2}$ cm/s, $x \sim 2$ cm) are in the range $X \sim 300$, $T \sim 200$. We first analyze the transition from the initial sharp concentration jump to the stationary profile; the solid lines in Fig. 1 are the solutions [Eq. (7)] [$C(x,t)/C_0$ vs X] for times $T = 100$, 200, 300, 400; the dashed lines are the corresponding stationary solutions [Eq. (9)]. This plot shows that an invariant shape, traveling at constant velocity V_s, is nearly achieved for time $T = 400$, about twice the experimental value ($T \sim 200$). Hence, the full transient solution [Eq. (7)] is required as expected from the above criterion ($T \gg T_c$, $T_c \sim 50$). Therefore, in a classical sedimentation experiment, the stationary profile is not generally reached before polydispersity effects become important and hence the S shape of the profile spreads as time goes on. Following the analysis of Davis *et al.*,[2] a fit of the broadening of this S shape profile with an error function [Eq. (2)] can be tested. The solid lines of Fig. 2 are the full transient solutions of Eq. (7) as in Fig. 1, for $T = 100$, 200, 300, the dashed lines are the error function [Eq. (2)] best fit to the transient solutions. Surprisingly, the agreement is quite perfect, for one reduced time ($T = 200$), but the apparent diffusion coefficient needed, $D_{ap} = 4.2aU(C_0)$, is nearly five times smaller than the above value [$D = 22aU(C_0)$]. Hence, it is straightforward to reconcile both data analysis by Davis *et al.* and Lee *et al.* Davies *et al.* used Eq. (2) and obtained D_{ap}. Lee *et al.* evaded the problem deriving the partial derivatives involved in the convection–diffusion equation [Eq. (1)] directly from their experimental data: In this way they determined the "correct" D. If we were to fit the experimental data of Fig. 3 of Lee *et al.*, for $t = 332$ s with an error function, we would find a value of $D_{ap} = 4aU(C_0)$. As a last comment, due to polydispersity, it is rather difficult to observe this stationary traveling profile in a sedimenting suspension. Observation of this front, as well as the determination of the dispersion coefficient, is easier in a fluidized bed where the profile is at steady state.[12]

We have shown that when self-sharpening and hydrodynamic dispersion are taken into account in the sedimentation of a monodisperse suspension of noncolloidal particles, the resulting sedimentation profile is a stationary front shape propagating with a constant velocity. The early stage of the transient regime has been analyzed in the framework of a Burger's equation and is found to provide a suitable means to measure the hydrodynamic coefficient.

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