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General Solutions of the Equations of Elasticity and Consolidation for a Porous Material[1]

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Equations of elasticity and consolidation for a porous elastic material containing a fluid have been previously established (1, 5).[3] General solutions of these equations for the isotropic case are developed, giving directly the displacement field or the stress field in analogy with the Boussinesq-Papkovitch solution and the stress functions of the theory of elasticity. General properties of the solutions also are examined and the viewpoint of eigenfunctions in consolidation problems is introduced.

1 Introduction

The theory of deformation of an elastic porous material containing a compressible fluid has been established by the author in previous publications. Reference (1) deals with the case of an isotropic material. In reference (5) the theory has been generalized to anisotropic materials. The model of a porous elastic body was suggested by Terzaghi (8) to represent the consolidation and settlement of fluid saturated soils. Hence the theory also has been referred to as a consolidation theory. Another category of problems covered by this theory is that of the flow of a compressible fluid in a porous material from the standpoint of determining the flow pattern of the fluid or the stresses set up by the fluid seepage through the elastic solid. Such a theory is applicable, for instance, to the problem associated with the loading process of a dam and in certain fundamental problems of petroleum geology.

Several problems of two-dimensional consolidation have been treated by the author (2, 3, 4). These papers make extensive use of the operational calculus and develop certain numerical methods appropriate to the problem.

The object of the present paper is to furnish general solutions of the equations for the isotropic case by means of functions satisfying the Laplace and the heat-conduction equation. The solutions developed in Section 3 are analogous to the Boussinesq-Papkovitch functions of the theory of elasticity (9, 10, 11). They yield directly the displacement field. Stress functions introduced in Section 4 express directly the stress field. In Section 5 general properties of the solutions are examined.

The concept of "consolidation mode" is introduced in Section 6, and it is pointed out that solutions expressed in this form are a consequence of a general property of relaxation phenomena (7).

2 The Elasticity and Consolidation Theory for a Porous Solid

In references (1) and (5) we have established a theory for the deformation of a porous elastic material containing a fluid. The equations obtained in the case of isotropy are reviewed briefly in the present section. The stress field of the porous material is denoted by

$$\left\{\begin{matrix} \sigma_{xx}+\sigma & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy}+\sigma & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}+\sigma \end{matrix}\right\} \qquad [1]$$

The $\sigma_{xx}\sigma_{xy}$, etc., components represent the forces acting on the solid part of the faces of a unit cube of bulk material while σ represents the force applied to the fluid part. Introducing the porosity f, the pressure p in the fluid is related to σ by

$$\sigma = -fp \qquad [2]$$

The average displacement vector of the solid has components $u_x u_y u_z$ and that of the fluid $V_x V_y V_z$. We introduce the following strain components for the solid

$$e_{xx} = \frac{\partial u_x}{\partial x} \quad e_{xy} = \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} \ldots \text{etc.} \qquad [3]$$

The only relevant strain component of the fluid is the dilation

$$\epsilon = \frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z} \qquad [4]$$

We also introduce the dilation of the solid

$$e = e_{xx} + e_{yy} + e_{zz} \qquad [5]$$

The stress-strain relations are

$$\left.\begin{aligned} \sigma_{xx} &= 2Ne_{xx} + Ae + Q\epsilon \\ \sigma_{yy} &= 2Ne_{yy} + Ae + Q\epsilon \\ \sigma_{zz} &= 2Ne_{zz} + Ae + Q\epsilon \\ \sigma_{yz} &= Ne_{yz} \\ \sigma_{zx} &= Ne_{zx} \\ \sigma_{xy} &= Ne_{xy} \\ \sigma &= Qe + R\epsilon \end{aligned}\right\} \qquad [6]$$

There are in this case four elastic constants, A, N, Q, and R.

If we neglect the body force, which is irrelevant to our problem, the stress field must satisfy the following equilibrium relations

$$\left.\begin{aligned} \frac{\partial(\sigma_{xx}+\sigma)}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} &= 0 \\ \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial(\sigma_{yy}+\sigma)}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} &= 0 \\ \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial(\sigma_{zz}+\sigma)}{\partial z} &= 0 \end{aligned}\right\} \qquad [7]$$

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The equations of flow for the fluid in the porous solid are written

$$\left.\begin{aligned} \frac{\partial\sigma}{\partial x} &= b\frac{\partial}{\partial t}(U_x - u_x) \\ \frac{\partial\sigma}{\partial y} &= b\frac{\partial}{\partial t}(U_y - u_y) \\ \frac{\partial\sigma}{\partial z} &= b\frac{\partial}{\partial t}(U_z - u_z) \end{aligned}\right\} \qquad [8]$$

Substituting Equations [6] into the equilibrium Equations [7] and the flow Equations [8] yields the six equations

$$\begin{aligned} N\nabla^2\bar{u} + (P - N + Q)\operatorname{grad} e + (Q + R)\operatorname{grad}\epsilon &= 0 \\ \operatorname{grad}(Qe + R\epsilon) &= b\frac{\partial}{\partial t}(\bar{U} - \bar{u}) \qquad [9] \end{aligned}$$

We have put

$$P = A + 2N \qquad [10]$$

The vectors $\bar{u}$ and $\bar{U}$ are the displacement of solid and fluid, respectively

$$\left.\begin{aligned} \bar{u} &= (u_x, u_y, u_z) \\ \bar{U} &= (U_x, U_y, U_z) \end{aligned}\right\} \qquad [11]$$

Also, by definition we have

$$\left.\begin{aligned} \epsilon &= \operatorname{div}\bar{u} \\ e &= \operatorname{div}\bar{U} \end{aligned}\right\} \qquad [11a]$$

We shall now formulate general solutions of Equations [9].

3 Solutions by Means of Displacement Functions

Since the relative displacement $\bar{U} - \bar{u}$ of the fluid is initially zero, the second Equation [9] shows that it must be irrotational; hence we put

$$\bar{U} - \bar{u} = \operatorname{grad}\varphi \qquad [12]$$

Then we introduce the new unknowns u_1 and φ in the differential Equations [9] by the substitution

$$\left.\begin{aligned} \bar{u} &= \bar{u}_1 - \frac{R + Q}{H}\operatorname{grad}\varphi \\ \bar{U} &= \bar{u}_1 + \frac{P + Q}{H}\operatorname{grad}\varphi \end{aligned}\right\} \qquad [13]$$

with $H = P + R + 2Q$ we find

$$\left.\begin{aligned} N\nabla^2 u_1 + (H - N)\operatorname{grad} e_1 &= 0 \\ (Q + R)e_1 + K\nabla^2\varphi &= b\frac{\partial\varphi}{\partial t} \end{aligned}\right\} \qquad [14]$$

with

$$e_1 = \operatorname{div}\bar{u}_1$$

$$K = \frac{PR - Q^2}{H} \qquad [15]$$

We note that $\bar{u}_1$, at the instant of loading, is equal to the displacement of the solid. Hence we may assume that $\varphi = 0$ at $t = 0$. The advantage of Equations [14] is that the first equation is the same as for the displacement vector in the theory of elasticity. They are solved by the introduction of the Boussinesq-Papkovitch functions (9, 10); namely, a scalar ψ_0 and a vector $\bar{\psi}$ satisfying Laplace's equation in Cartesian co-ordinates

$$\nabla^2\psi_0 = \nabla^2\bar{\psi} = 0 \qquad [16]$$

We may then write for a complete solution of the first Equation [14] the expression

$$\bar{u}_1 = -\operatorname{grad}(\psi_0 + \bar{r}\cdot\bar{\psi}) + \frac{2H}{H - N}\bar{\psi} \qquad [17]$$

where

$$\bar{r} = (x, y, z)$$

Because of the property which is derived from the first Equation [14] that

$$\nabla^2 e_1 = 0 \qquad [18]$$

the solution of the second Equation [14] is

$$\varphi = \frac{Q + R}{b}\int_0^t e_1 dt + \phi \qquad [19]$$

where ϕ now satisfies the heat-conduction equation

$$K\nabla^2\phi = b\frac{\partial\phi}{\partial t} \qquad [20]$$

We may write e_1, in terms of the functions $\bar{\psi}$

$$e_1 = \operatorname{div} u_1 = \frac{2N}{H - N}\operatorname{div}\bar{\psi} \qquad [21]$$

Then using Equations [13], [19], and [21], the solid displacement $\bar{u}$ is given by the expression

$$\begin{aligned} \bar{u} = &-\operatorname{grad}(\psi_0 + \bar{r}_0\bar{\psi}) + \frac{2H}{H - N}\bar{\psi} \\ &- \frac{2N(Q + R)^2}{bH(H - N)}\int_0^t \operatorname{grad}\operatorname{div}\bar{\psi}dt - \frac{Q + R}{H}\operatorname{grad}\phi \qquad [22] \end{aligned}$$

A similar expression may be written for $\bar{U}$.

Completeness of the solution, Equation [17], was established by Mindlin (11). Hence Equation [22] is also a complete solution of Equation [9] in terms of a scalar ψ_0 and a vector $\bar{\psi}$ satisfying Laplace's Equation [16] and a scalar ϕ satisfying the heat-conduction Equation [20].[4] Since the Boussinesq-Papkovitch solution is the same as that used in the theory of elasticity it appears therefore that from a mathematical standpoint what distinguishes the consolidation problem from an elasticity problem is the addition of the function ϕ satisfying a heat-conduction-type equation.

The settlement is found by subtracting from $\bar{u}$ the value of $\bar{u}$ at the instant immediately after loading found by putting $t = 0$ and $\phi = 0$ in Expression [22].

Substitution of the values of $\bar{u}$ and $\bar{U}$ in Expression [6] yields the stresses in the solid and the pressure in the fluid.

Particular problems may be solved conveniently by the operational methods as exemplified in references (2, 3, 4). The foregoing solutions have the advantage of mathematical symmetry. However, in some problems it may be advantageous to introduce a different set of variables. We write the stress-strain relations as

$$\left.\begin{aligned} \sigma_{\mu\mu} + \sigma &= 2Ne_{\mu\mu} + Se - \alpha p \\ \sigma_{\mu\nu} &= Ne_{\mu\nu} \quad (\mu \neq \nu) \end{aligned}\right\} \qquad [23]$$

with

$$\alpha = \frac{Q + R}{R}f \quad S = A - \frac{Q^2}{R}$$

[4] It will be noted that the grad operator which was dropped in the second Equation [14] is re-established in Expression [22].

The coefficients N and S are Lamé constants in the absence of fluid pressure p. We also introduce a constant

$$M = R/f^2 \qquad [24]$$

and a variable

$$\zeta = -f(\epsilon - e) = \frac{1}{M} p + \alpha e \qquad [25]$$

The variable ζ is the change of fluid content in volume per unit volume. It is interesting to note the relation between ζ and φ. We find

$$\zeta = -f\nabla^2\varphi \qquad [26]$$

Also, we introduce a permeability

$$k = f^2/b \qquad [27]$$

such that

$$-k \operatorname{grad} p = f \frac{\partial}{\partial t} (\bar{U} - \bar{u}) \qquad [28]$$

These variables were used originally in reference (1).

Introducing the variable ζ instead of σ, Equations [9] may be written

$$N\nabla^2\bar{u} + (N + S + \alpha^2 M) \operatorname{grad} e - \alpha M \operatorname{grad} \zeta = 0$$

$$kK_1\nabla^2\zeta = \frac{\partial\zeta}{\partial t} \qquad [29]$$

with

$$K_1 = \frac{(2N + S)M}{2N + S + \alpha^2 M} = \frac{PR - Q^2}{Hf^2} = \frac{K}{f^2}$$

The second equation is independent of the deformation field and expresses the remarkable property that the fluid content ζ satisfies the diffusion equation. This property is also expressed below by Equation [53]. Proceeding as for Equations [14] it is seen that the general solution for $\bar{u}$ is

$$\bar{u} = -\operatorname{grad} (\psi_0 + \bar{r}\bar{\psi}) + \frac{2(2N + S + \alpha^2 M)}{N + S + \alpha^2 M} \bar{\psi} \qquad [30]$$

with $\bar{\psi}$ satisfying

$$\nabla^2\bar{\psi} = 0 \qquad [31]$$

and

$$(2N + S + \alpha^2 M)\nabla^2\psi_0 + \alpha M\zeta = 0 \qquad [31a]$$

Actually, the last equation contains a grad operator. However, it can be dropped since it adds to ψ_0 a quadratic function of the coordinates which may be included in the Solution [30] by adding a linear function of the co-ordinates in the value of $\bar{\psi}$.

4 Solution by Means of Stress Functions

In some problems the emphasis is on the calculation of the stress rather than the displacement. It is therefore useful to investigate the possibility of expressing solutions by means of stress functions rather than displacement functions. We consider the problem of two-dimensional strain where the strain components $e_{zz}e_{yz}$ and e_{zx} all vanish and the displacement vectors $\bar{u}$ and $\bar{U}$ have only (x, y) components independent of z. The stress-strain relations are

$$\left.\begin{aligned} \sigma_{xx} &= 2Ne_{xx} + Ae + Q\epsilon \\ \sigma_{yy} &= 2Ne_{yy} + Ae + Q\epsilon \\ \sigma_{xy} &= Ne_{xy} \\ \sigma &= Qe + R\epsilon \end{aligned}\right\} \qquad [32]$$

In addition

$$\sigma_{zz} = Ae + Q\epsilon \qquad [33]$$

All quantities are independent of z.

The equations of equilibrium, Equations [7], may be satisfied by introducing a stress function F

$$\left.\begin{aligned} \sigma_{xx} + \sigma &= \frac{\partial^2 F}{\partial y^2} \\ \sigma_{yy} + \sigma &= \frac{\partial^2 F}{\partial x^2} \\ \sigma_{xy} &= -\frac{\partial^2 F}{\partial x \partial y} \end{aligned}\right\} \qquad [34]$$

This represents the total stress field in the bulk material. If we eliminate ϵ between Equations [32] we find

$$\left.\begin{aligned} \sigma_{xx} &= 2Ne_{xx} + Se + \frac{Q}{R}\sigma \\ \sigma_{yy} &= 2Ne_{yy} + Se + \frac{Q}{R}\sigma \\ \sigma_{xy} &= Ne_{xy} \end{aligned}\right\} \qquad [35]$$

with

$$S = A - \frac{Q^2}{R} \qquad [36]$$

Introducing Equations [34] this becomes

$$\left.\begin{aligned} \frac{\partial^2 F}{\partial y^2} &= 2Ne_{xx} + Se + \frac{Q + R}{R}\sigma \\ \frac{\partial^2 F}{\partial x^2} &= 2Ne_{yy} + Se + \frac{Q + R}{R}\sigma \\ \frac{\partial^2 F}{\partial x \partial y} &= -Ne_{xy} \end{aligned}\right\} \qquad [37]$$

Adding the two first Equations [37] and taking into account that

$$e = e_{xx} + e_{yy} \qquad [38]$$

we find

$$\frac{1}{2}\nabla^2 F = (N + S)e + \frac{(Q + R)}{R}\sigma \qquad [39]$$

We now introduce the well-known compatibility relation of the strain tensor. This relation is the following identity

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} - \frac{\partial^2 e_{xy}}{\partial x \partial y} = 0 \qquad [40]$$

It may be used to eliminate the strain components from Equations [37] if we add these three equations after applying to each the proper differential operator. This gives

$$\nabla^4 F = S\nabla^2 e + \frac{Q + R}{R}\nabla^2\sigma \qquad [41]$$

Finally, eliminating e between Equations [39] and [41] gives a relation between F and σ

$$(PR - Q^2)\nabla^4 F = 2N(Q + R)\nabla^2\sigma \qquad [42]$$

In order to solve the problem we need another equation with F

and σ. This may be obtained from the second Equation [9] which expresses Darcy's law. By forming the divergence we have

$$\nabla^2(Qe + R\epsilon) = b\frac{\partial}{\partial t}(\epsilon - e) \quad [43]$$

The quantities e and ϵ may be expressed in terms of F and σ from the last Equation [32] and Equation [39]. Introducing these values in Equation [43] it becomes

$$(PR - Q^2 - NR)\nabla^2\sigma = b(H - N)\frac{\partial\sigma}{\partial t} - \frac{b}{2}(Q + R)\frac{\partial}{\partial t}\nabla^2 F \quad [44]$$

This last equation may be given a more convenient form by combining with Equation [42]. If we multiply Equation [42] by $Q + R$ and Equation [44] by $-H$, then add these equations, we obtain after dividing by H

$$\left(K\nabla^2 - b\frac{\partial}{\partial t}\right)[(Q + R)\nabla^2 F - 2(H - N)\sigma] = 0 \quad [45]$$

Equations [42] and [45] constitute a system of two simultaneous equations for the stress function and the fluid pressure.

A primary inconvenience in using the stress function is that boundary conditions involving displacements require evaluation of these displacements from the stress. Also, it is well known that if the body contains cavities, single-valued stress functions may lead to multivalued displacements or dislocations. The condition that displacements be single-valued requires the use of an additional condition (6). In such cases it will generally be simpler to solve the problem by means of displacement functions. However, when the boundary conditions are given in terms of stresses and if the field contains no cavities, the problem should be conveniently solved by the use of stress functions.

It is interesting to note that elimination of F between Equations [42] and [45] leads to

$$K\nabla^4\sigma = b\frac{\partial}{\partial t}\nabla^2\sigma \quad [46]$$

The fluid pressure is therefore the sum of two functions satisfying, respectively, the two equations

$$\left.\begin{aligned} K\nabla^2\sigma &= b\frac{\partial\sigma}{\partial t} \\ \nabla^2\sigma &= 0 \end{aligned}\right\} \quad [47]$$

Also, elimination of σ between the same Equations [42] and [45] yields

$$K\nabla^6 F = b\frac{\partial}{\partial t}\nabla^4 F \quad [48]$$

Similarly, F is the sum of two functions satisfying, respectively, the two equations

$$\left.\begin{aligned} K\nabla^2 F &= b\frac{\partial F}{\partial t} \\ \nabla^4 F &= 0 \end{aligned}\right\} \quad [49]$$

5 General Properties of Solutions

It is possible to derive certain general properties of the solutions. Starting from Expression [13] for $\bar{u}$ and $\bar{U}$ we find by applying the divergence operator

$$\left.\begin{aligned} e &= e_1 - \frac{R + Q}{H}\nabla^2\varphi \\ \epsilon &= e_1 + \frac{P + Q}{H}\nabla^2\varphi \end{aligned}\right\} \quad [50]$$

Elimination of $\nabla^2\varphi$ or e_1 gives, respectively

$$\left.\begin{aligned} (P + Q)e + (R + Q)\epsilon &= He_1 \\ \epsilon - e &= \frac{P + R}{H}\nabla^2\varphi \end{aligned}\right\} \quad [51]$$

Applying the Laplace operator to the first equation and taking Equation [18] into account

$$\nabla^2[(P + Q)e + (R + Q)\epsilon] = 0 \quad [52]$$

Applying the operator

$$K\nabla^2 - b\frac{\partial}{\partial t}$$

to the second Equation [51] and taking Equations [14] and [18] into account

$$\left(K\nabla^2 - b\frac{\partial}{\partial t}\right)(\epsilon - e) = 0 \quad [53]$$

Similar relations exist for the stresses in the solid. To show this let us introduce the hydrostatic stress σ_h in the solid

$$\sigma_h = \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) \quad [54]$$

From the stress-strain Relation [6] we derive

$$\left.\begin{aligned} \sigma_h &= Be + Q\epsilon \\ \sigma &= Qe + R\epsilon \end{aligned}\right\} \quad [55]$$

with

$$B = P - \frac{4}{3}N \quad [56]$$

Solving Equation [55] for e and ϵ and substituting in Relations [52] and [53] yields

$$\left.\begin{aligned} &\nabla^2\left\{\sigma_h(PR - Q^2) + \sigma\left[PR - Q^2 - \frac{4}{3}N(Q + R)\right]\right\} = 0 \\ &\left(K\nabla^2 - b\frac{\partial}{\partial t}\right)[\sigma_h(Q + R) - \sigma(B + Q)] = 0 \end{aligned}\right\} \quad [57]$$

Finally, by applying the operator

$$\nabla^2\left(K\nabla^2 - b\frac{\partial}{\partial t}\right)$$

to Expression [50] and taking into account Equations [14] and [18], we find

$$\left.\begin{aligned} \nabla^2\left(K\nabla^2 - b\frac{\partial}{\partial t}\right)e &= 0 \\ \nabla^2\left(K\nabla^2 - b\frac{\partial}{\partial t}\right)\epsilon &= 0 \end{aligned}\right\} \quad [58]$$

Also, from Equations [55] and [58]

$$\left.\begin{aligned}\nabla^2\left(K\nabla^2 - b\frac{\partial}{\partial t}\right)\sigma_h &= 0\\ \nabla^2\left(K\nabla^2 - b\frac{\partial}{\partial t}\right)\sigma &= 0\end{aligned}\right\}\quad [59]$$

It is interesting to note that Equation [45] may be derived from the second Equation [57].

We write

$$X = (B + Q)\sigma - (R + Q)\sigma_h \quad [60]$$

and substitute the value of σ_h in terms of $\nabla^2 F$ and σ by the relation

$$\left.\begin{aligned}D\nabla^2 F &= 2\sigma_h\left(D + \frac{1}{3}NR\right) + 2\sigma\left(D - \frac{1}{3}QN\right)\\ &\text{with}\\ D &= BR - Q^2 = PR - Q^2 - \frac{4}{3}NR\end{aligned}\right\}\quad [61]$$

We find

$$2X\left(D + \frac{1}{3}NR\right) = D[2\sigma(H - N) - (R + Q)\nabla^2 F] \quad [62]$$

Applying the operator

$$K\nabla^2 - b\frac{\partial}{\partial t}$$

to the equation and taking into account that, from Equation [57]

$$\left(K\nabla^2 - b\frac{\partial}{\partial t}\right)X = 0 \quad [63]$$

we find Equation [45].

One interesting particular case of the general solution is when the displacement vector in the solid is the gradient of a scalar, i.e., when the rotation vanishes. Such a case occurs if we put

$$\bar{\psi} = 0 \quad [64]$$

in Equation [22]. The solid and fluid displacements are

$$\left.\begin{aligned}\bar{u} &= -\text{grad}\left[\psi_0 + \frac{Q + R}{H}\phi\right]\\ \bar{U} &= -\text{grad}\left[\psi_0 - \frac{P + Q}{H}\phi\right]\end{aligned}\right\}\quad [65]$$

with

$$\left.\begin{aligned}\nabla^2\psi_0 &= 0\\ \left(K\nabla^2 - b\frac{\partial}{\partial t}\right)\phi &= 0\end{aligned}\right\}\quad [66]$$

We have

$$\left.\begin{aligned}e &= \text{div}\,\bar{u} = -\frac{Q + R}{H}\nabla^2\phi\\ \epsilon &= \text{div}\,\bar{U} = \frac{P + Q}{H}\nabla^2\phi\end{aligned}\right\}\quad [67]$$

The fluid pressure is given by the last Equation [6]

$$\sigma = Qe + R\epsilon \quad [68]$$

Substituting Equation [67]

$$\sigma = K\nabla^2\phi \quad [69]$$

Because of Equation [66] we may write

$$\left(K\nabla^2 - b\frac{\partial}{\partial t}\right)\sigma = 0 \quad [70]$$

The pore pressure in this case satisfies the heat-conduction equation.

By substituting Equations [65] and [69] into Expression [6] for the stress we find

$$\left.\begin{aligned}\sigma_{xx} &= -2N\frac{\partial^2}{\partial x^2}\left[\psi_0 + \frac{R + Q}{H}\phi\right] - \left[\frac{2N(Q + R)}{PR - Q^2} - 1\right]\sigma\\ \sigma_{xy} &= -2N\frac{\partial^2}{\partial x\partial y}\left[\psi_0 + \frac{R + Q}{H}\phi\right], \text{ etc.}\end{aligned}\right.\quad [71]$$

The function ϕ may be determined from σ by taking into account Equations [66] and [69], i.e.

$$\sigma = b\frac{\partial\phi}{\partial t} \quad [72]$$

Since $\phi = 0$ at the instant of loading we have

$$\phi = \frac{1}{b}\int_0^t \sigma\, dt \quad [73]$$

These expressions are particularly useful in solving problems with spherical symmetry or cylindrical problems with circular symmetry.

For instance, the problem of loading of a spherical cavity in an infinite porous material is easily solved by putting

$$\psi_0 = \frac{C}{r} \quad [74]$$

with

$$r = \sqrt{x^2 + y^2 + z^2}$$

while σ is given by the well-known expression for heat conduction with spherical symmetry. Similarly, for a cylindrical cavity we put

$$\psi_0 = C\log r \quad [75]$$

with $r = \sqrt{x^2 + y^2}$ and the corresponding heat-conduction expressions for σ.

More general irrotational solutions are obtained by adding to the foregoing the trivial case of uniform hydrostatic pressure in the pores along with a state of uniform isotropic stress in the solid. This amounts to putting $\bar{\psi}$ equal to the gradient of a scalar. With these additional solutions we can solve the case of the sphere or cylinder with finite thickness.

6 Eigenfunctions and Modes of Consolidation

Another procedure in the solution of consolidation problems is to introduce eigenfunctions and the concept of consolidation modes. This is best illustrated by an example:

Consider the one-dimensional consolidation problem when a column of height h is under a vertical load γ per unit area. The load is applied through a porous slab so that the fluid pressure at the top boundary is zero. The z-co-ordinate is along the vertical direction, the bottom of the column being at $z = 0$ and the top at $z = h$.

Equation [9] becomes in this case

$$\left.\begin{aligned} (P+Q)\frac{\partial^2 u_z}{\partial z^2} + (Q+R)\frac{\partial^2 U_z}{\partial z^2} &= 0 \\ Q\frac{\partial^2 u_z}{\partial z^2} + R\frac{\partial^2 U_z}{\partial z^2} &= b\frac{\partial}{\partial t}(U_z - u_z) \end{aligned}\right\} \ldots\ldots [76]$$

These equations are solved by Expressions [13] which in the present case take the form

$$\left.\begin{aligned} u_z &= u_1 - \frac{R+Q}{H}\frac{\partial \varphi}{\partial z} \\ U_z &= u_1 + \frac{P+Q}{H}\frac{\partial \varphi}{\partial z} \end{aligned}\right\} \ldots\ldots [77]$$

Substituting in Equations [76] leads to

$$\frac{\partial^2 u}{\partial z^2} = 0$$

$$(Q+R)\frac{\partial^2 u_1}{\partial z^2} + K\frac{\partial^2 \varphi}{\partial z^2} = b\frac{\partial \varphi}{\partial t} \ldots\ldots [78]$$

In order to introduce the boundary condition let us express the stress

$$\left.\begin{aligned} \sigma_{zz} &= (P+Q)\frac{\partial u_1}{\partial z} - K\frac{\partial^2 \varphi}{\partial z^2} \\ \sigma &= (Q+R)\frac{\partial u_1}{\partial z} + K\frac{\partial^2 \varphi}{\partial z^2} \end{aligned}\right\} \ldots\ldots [79]$$

We separate the solution into a part independent of time and a variable one. We consider u_1 independent of time and we put

$$\varphi = \varphi_1 + \phi \ldots\ldots [80]$$

where ϕ is the time-dependent part of φ.

Because of Equation [78] and the boundary condition $\sigma = 0$ at $z = h$ we find

$$\frac{\partial u_1}{\partial z} = C_1 \quad \frac{\partial^2 \varphi_1}{\partial z^2} = C_2 \ldots\ldots [81]$$

with

$$\left.\begin{aligned} (Q+R)C_1 + KC_2 &= 0 \\ (P+Q)C_1 - KC_2 &= -\gamma \end{aligned}\right\} \ldots\ldots [82]$$

Hence

$$\left.\begin{aligned} C_1 &= -\gamma/H \\ C_2 &= \frac{\gamma(Q+R)}{PR - Q^2} \end{aligned}\right\} \ldots\ldots [83]$$

The time-dependent part ϕ satisfies the equation

$$\frac{\partial^2 \phi}{\partial z^2} = \frac{1}{a}\frac{\partial \phi}{\partial t} \ldots\ldots [84]$$

with $a = b/K$. Solutions of this equation are

$$\phi = \cos z\sqrt{\frac{\alpha}{a}}\, e^{-\alpha t} \ldots\ldots [85]$$

The boundary condition that $u_z = U_z = 0$ at $z = 0$ is not disturbed since $\partial\phi/\partial z = 0$ at that point. The stresses at the upper boundary are not disturbed if we have $\partial^2\phi/\partial z^2 = 0$ at $z = h$; hence

$$h\sqrt{\frac{\alpha}{a}} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \ldots \left(\frac{2n+1}{2}\right)\pi \ldots\ldots [86]$$

These lead to the eigenfunctions

$$\phi_n = \cos z\sqrt{\frac{\alpha_n}{a}}\, e^{-\alpha_n t} \ldots\ldots [87]$$

with the characteristic values for α

$$\alpha_n = \left(\frac{2n+1}{2}\right)^2 \frac{\pi^2 a}{h^2} \ldots\ldots [88]$$

The consolidation problem is then solved by a series of eigenfunctions

$$\phi = \sum^{n} A_n \phi_n \ldots\ldots [89]$$

when A_n are Fourier coefficients determined by the initial condition

$$\varphi = 0 \text{ for } t = 0$$

or

$$\frac{\partial^2 \phi}{\partial z^2} = -\frac{\partial^2 \varphi_1}{\partial z^2} = -C_2 \ldots\ldots [90]$$

The time-dependent part of the deformation is then expressed as the sum of an infinite number of "modes of consolidation," such that for each of these functions the displacements are proportional to the same decreasing exponential $e^{-\alpha_n t}$. This procedure may be generalized to the three-dimensional case. The existence of consolidation modes is a consequence of a general property of relaxation phenomena as shown by the author in reference (7).

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