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Film flows down a fiber: Modeling and influence of streamwise viscous diffusion

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Abstract. A two-equation model is formulated in terms of two coupled evolution equations for the film thickness h and the local flow rate q within the framework of lubrication theory. Consistency is achieved up to first order in the film parameter ϵ and streamwise diffusion effects are accounted for. The evolution equation obtained by Craster and Matar [1] is recovered in the appropriate limit. Comparisons to the experimental results by [2] and [3] show good agreement in the linear and nonlinear regimes. Second-order viscous diffusion terms are found to potentially enhance the speed and amplitude of nonlinear waves triggered by the Rayleigh-Plateau instability mechanism. Time-dependent computations of the spatial evolution of the film reveal a strong influence of streamwise diffusion on the dynamics of the flow and the wave selection process.

Besides their importance for engineering applications (*e.g.* evaporators or chemical reactors), the interest of falling liquid films mainly stems from the fact that their evolution is tractable to thorough theoretical analysis. We consider here the evolution of axisymmetric waves on a liquid film falling down a vertical fiber. This problem has attracted numerous recent works [1–5], a situation that is probably due to the large panel of wave regimes observable in the experiments, *e.g.* noise-driven dynamics characterized by solitary wave interactions, regular "drop-like" wave patterns, spatial modulations of multi-hump solitary-like wavetrains excited by a periodic forcing at inlet, bi-periodic wave patterns made of a repetition of merging and wave radiations, etc. This variety of nonlinear phenomena arises from the non-trivial interaction of the Rayleigh-Plateau instability mechanism in cylindrical geometry, and the classical Kapitza instability of falling films.

Despite a continuous modeling effort, low-dimensional models available in literature either neglect inertial effects [1,2], or underestimate streamwise viscous terms that modify the wave dispersion [4], or consider only small curvature effects [5]. There is therefore a crucial need for a low-dimensional formulation that overcomes these limitations. We develop here a weighted residual-method based on the long-wave assumption – evolution in space and time is slow which justifies the introduction of a formal film parameter ϵ.

The thickness of the layer and the fiber radius are denoted by h and R respectively. x, r and t are the axial, radial and time coordinates. The fluid properties, viscosity μ, density ρ and surface tension σ are all assumed to remain constant. The basic set of equations is made dimensionless with reference to the steady and parallel solution of constant thickness h_{N} that is observed at inlet and corresponds to the balance of viscous drag and gravity acceleration.

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The distribution of the axial velocity is then given by

$$u_x(x,t) = \frac{g}{\nu}\left[\frac{1}{2}(R+h)^2 \ln\left(\frac{r}{R}\right) - \frac{1}{4}(r^2 - R^2)\right], \tag{1}$$

where $\nu = \mu/\rho$ is the kinematic viscosity. The corresponding flow rate $q \equiv R^{-1}\int_R^{R+h} u_x r\,\mathrm{d}r$ reads

$$q_{\mathrm{N}} \equiv \frac{g}{3\nu} h^3 \phi(h/R)\,, \tag{2}$$

where $\phi(\alpha) \equiv 3\left(4(\alpha+1)^4\log(\alpha+1) - \alpha(\alpha+2)(3\alpha(\alpha+2)+2)\right)(16\alpha^3)^{-1}$. Setting $h = h_{\mathrm{N}}\tilde{h}$, $x = \kappa h_{\mathrm{N}}\tilde{x}$, and $t = T\tilde{t}$. The ratio κ of the length scales in the cross-stream and streamwise coordinates is adjusted such that the gravity acceleration ρg equilibrates the pressure gradient induced by the main contribution to the axial curvature gradient $\sigma\partial_{x^3}h$. Thus $\kappa = \left[\sigma/(\rho g h_{\mathrm{N}}^2)\right]^3 = (l_c/h_{\mathrm{N}})^{2/3}$ where $l_c = \sqrt{\sigma/(\rho g)}$ is the capillary length. The time scale $T = \kappa h_{\mathrm{N}}^2/(3q_{\mathrm{N}})$ is one third of the advection time.

As the instability of a film of uniform thickness results either from the capillary pressure induced by azimuthal curvature (Rayleigh-Plateau instability mode) or from inertia (Kapitza instability mode), an accurate modeling of the flow requires to take into account these effects. Besides the growth of the waves is arrested by the capillary pressure induced by axial curvature. Finally, the stabilizing streamwise viscous effects must be accounted for. We develop a systematic strategy taking into account all leading order contributions from the above cited physical effects. We assume a slow evolution in space and time $\partial_{x,t} \sim \epsilon$ and small deviations of the velocity field from the uniform-thickness solution (1) so that the the velocity field is slaved to a limited number of variables such as the film thickness h or the local flow rate q. This assumption enables to formulate a system of one-dimensional equations through elimination of the cross-stream variable. The procedure can be decomposed into three steps: (i) formulation of boundary-layer like equations; (ii) expansion of the velocity profile on a closed set of trial functions; (iii) a weighted residual method enables to drastically reduce the required computations. The obtained model consists in two evolution equations for the film thickness h and the flow rate q [6].

$$\partial_t h = -\frac{1}{1+\tilde{\alpha}h}\partial_x q,$$

$$\delta\partial_t q = \delta\left[-F(\tilde{\alpha}h)\frac{q}{h}\partial_x q + G(\tilde{\alpha}h)\frac{q^2}{h^2}\partial_x h\right] + \frac{I(\tilde{\alpha}h)}{\phi(\tilde{\alpha})}\left[-\frac{3\phi(\tilde{\alpha})}{\phi(\tilde{\alpha}h)}\frac{q}{h^2}\right.$$

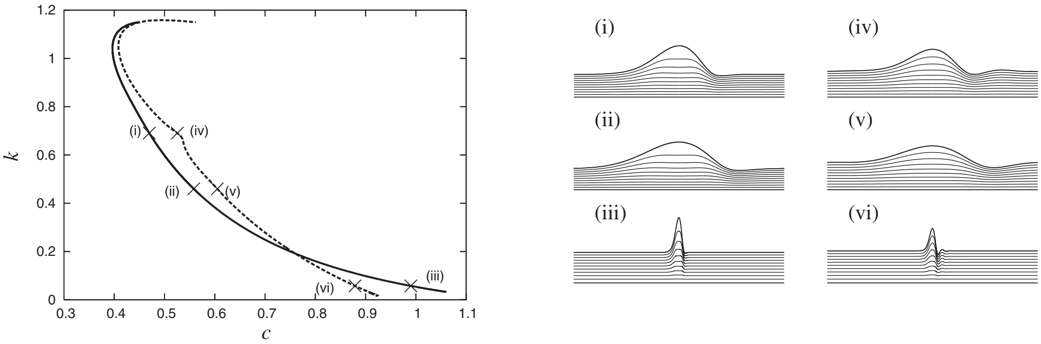
$$\left. + h\left\{1 + \partial_{xxx}h + \frac{\beta}{(1+\tilde{\alpha}h)^2}\partial_x h - \frac{1}{2}\partial_x\left(\frac{\tilde{\alpha}}{1+\tilde{\alpha}h}(\partial_x h)^2\right)\right\}\right] \tag{3a}$$

$$+ \eta\left[J(\tilde{\alpha}h)\frac{q}{h^2}(\partial_x h)^2 - K(\tilde{\alpha}h)\frac{\partial_x q \partial_x h}{h} - L(\tilde{\alpha}h)\frac{q}{h}\partial_{xx}h + M(\tilde{\alpha}h)\partial_{xx}q\right], \tag{3b}$$

where F, G, I, J, K, L, and M are functions of the aspect ratio $\tilde{\alpha} = h_{\mathrm{N}}/R$, $\delta \equiv h_{\mathrm{N}}^3\phi(\alpha h_{\mathrm{N}})/\kappa$ is a reduced Reynolds number, $\eta \equiv 1/\kappa^2 = (\bar{h}_{\mathrm{N}}/l_c)^{4/3}$ is a streamwise viscous diffusion parameter and $\beta = \tilde{\alpha}^2/\eta$ compares azimuthal and axial surface tension effects. Equation (3a) is the (exact) dimensionless mass balance whereas (3b) is the streamwise momentum averaged across the film. Contrary to the Trifonov model [4,7], (3b) is coherent up to order ϵ and accounts for viscous diffusion effects in the axial direction (last row of (3b)). The obtained formulation is not limited to small aspect ratios as for Roberts and Li model [5]. Besides in the limit of negligible inertia and streamwise viscous effects ($\delta \to 0$) and ($\eta \to 0$), the Craster and Matar equation [1] is recovered:

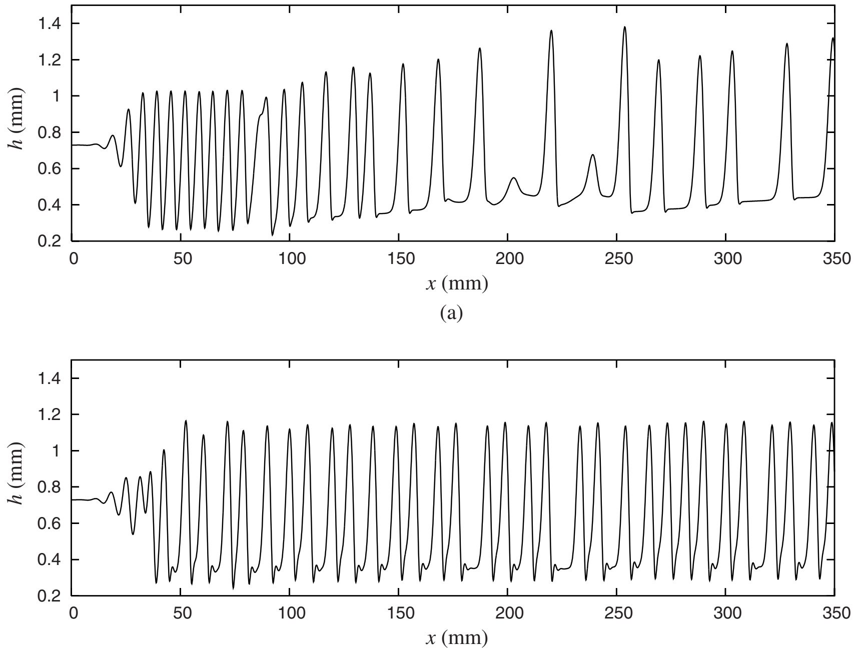
$$\partial_t\left(h + \frac{\tilde{\alpha}}{2}h^2\right) + \partial_x\left[\frac{h^3}{3}\frac{\phi(\tilde{\alpha}h)}{\phi(\tilde{\alpha})}\left(1 + \frac{\beta}{(1+\tilde{\alpha}h)^2}\partial_x h + \partial_{xxx}h\right)\right] = 0\,. \tag{4}$$

Linear stability analysis of the dispersion relation of model (3b) accurately recovers all feature of the Orr-Sommerfeld analysis of the basic linearized equations. In particular, the prediction of



(a) (b)

Fig. 1. (a) Speed c of traveling-wave solutions as function of k. Parameters correspond to castor oil ($\Gamma = 0.45$), $q_{\mathrm{N}} = 10$ mg/s and $R = 0.5$ ($\delta = 0.01$, $\eta = 0.22$, $\beta = 6.1$). Solid and dashed lines refer to (3b) and to the CM equation (4) respectively. (b) Wave profiles and streamlines in the moving frame of reference for solutions indicated by crosses in panel a; left: solutions to (3b), right: solutions to (4).

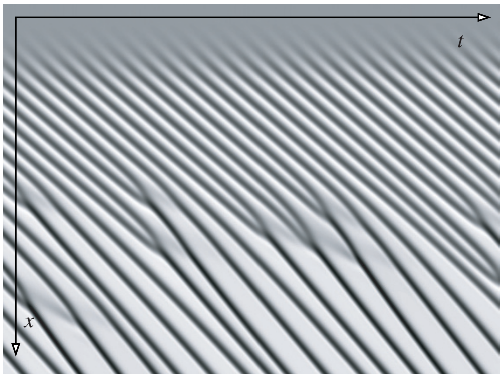


(b)

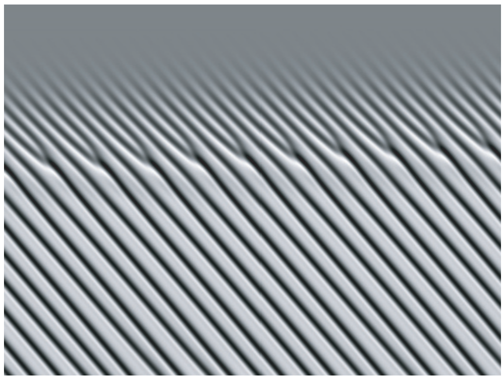
Fig. 2. Computed film thicknesses as function of the distance from inlet. Parameters correspond to the regime 'a' reported in Ref. [2]: $q_{\mathrm{N}} = 21$ mg/s, $R = 0.25\,$mm, $\nu = 440 \cdot 10^{-6}\,\mathrm{m}^2/\mathrm{s}$, $\sigma = 31 \cdot 10^{-3}\mathrm{N/m}$ ($\delta = 0.05$, $\beta = 28.7$, $\eta = 0.30$). (a) Solutions to model (3b); (b) CM equation (4).

the onset of an absolute instability is in excellent agreement with both Orr–Sommerfeld analysis and experimental data [3]. Traveling-wave solutions to the approximate model compare well to the experimental profiles [8]. Interestingly, a non-trivial amplification of the wave amplitude and wave speed by the second-order streamwise effects is observed at small Reynolds numbers (see figure 1).

Figure 2 presents time-dependent computations of the solutions to system (3b) and to the CM evolution equation (4) for a creeping flow ($\delta = 0.05$). Simulation relative to the two-equation model (3b) shows that the primary wavetrain is disorganized downstream by a secondary instability. Intermittent coalescence events widen the spacing between waves. The waves become more and more localized. At final stage, pulse-like solitary waves are separated by portions of flat films of close but irregular thicknesses. Amplitudes and distances between pulses are in



(a)



(b)

Fig. 3. Spatio-temporal diagrams corresponding to an experiment by Duprat et al. [3]. Horizontal and vertical axis correspond to space and time respectively. (a) solution to the second-order model (3b); (b) solution to the Trifonov model. Parameters correspond to experimental conditions [3, figure 2, silicon oil v50 and $R = 0.32\,\mathrm{mm}$]: $q_\mathrm{N} = 24\,\mathrm{mg/s}$ ($\delta = 0.3$, $\beta = 9.3$, $\eta = 0.19$). Light (dark) regions correspond to small (large) elevations. Vertical and horizontal ranges are 4 s and 10 cm.

reasonable agreement with the experimental observations (maximum film thickness $h_\mathrm{max}^\mathrm{expt} = 1.47\,\mathrm{mm}$ and approximately 30 mm between pulses). Time-dependent computations of the CM equation (4) show a radically different dynamics. The average separation between succeeding fronts, around 10 mm, is three times smaller than reported in experiments. The waves tend to group themselves in 'bounded states' of two or three pulses as observed by Craster and Matar [1].

When inertia is no more negligible (cf. figure 3), system (3b) still gives results in reasonable agreement with experimental observations. As in the experiment, the disorganization process of the primary wavetrain looks irregular and is probably promoted by a sideband instability. Simulation of the Trifonov model shows a different scenario. The wavetrain that emerges from the primary instability undergoes a subharmonic instability that doubles its frequency. No further bifurcations are observable downstream.

To conclude, a convincing agreement has been obtained with the experiments by Kliakhandler et al. [2] and by Duprat et al. [3,8]. Our simulations of the spatial evolution of the flow reveal that the wave selection observed with the CM equation do not correspond to the experimental observations. Therefore, the CM equation should be used with caution and certainly not to describe the spatio-temporal dynamics. The Trifonov model suffers from similar limitations. The origin of these inaccuracies can be found in the neglect of the streamwise viscous diffusion effects that are taken into account in the present formulation.

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